Bogoliubov Laboratory of Theoretical Physics
JOINT INSTITUTE FOR NUCLEAR RESEARCH
141980 Dubna (Moscow region), Russia

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REPRESENTATIONS FOR THREE-BODY T–MATRIX ON UNPHYSICAL SHEETS[1][2]

A.K.Motovilov[3]

Abstract

Explicit representations are formulated for the Faddeev components of three-body T-matrix continued analytically on unphysical sheets of the energy Riemann surface. According to the representations, the T–matrix on unphysical sheets is obviously expressed in terms of its components taken on the physical sheet only. The representations for T–matrix are used then to construct similar representations for analytical continuation of three-body scattering matrices and resolvent. Domains on unphysical sheets are described where the representations obtained can be applied.

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[3] E–mail: MOTOVILV@THSUN1.JINR.DUBNA.SU

1. INTRODUCTION

Resonances are one of the most interesting phenomena in scattering processes. The problem of defining and studying resonances in quantum mechanics is payed a lot of attention in physical and mathematical literature. In recent years, the investigations of resonances in few-particle systems attract a growing attention. The role of such resonances is well known in physics of nuclear reactions and astrophysics.

Developing methods for studying resonances has a long history beginning from the paper by G.Gamow [1]. In this paper devoted to description of α–decay, it was discussed for the first time a relation of resonance states to complex poles of the scattering matrix (it should be noted however that complex frequencies were considered much earlier, e.g. by J.J.Thomson in 1884). For spherically symmetrical potentials, the interpretation of resonances in two-body problems as poles of an analytic continuation of the scattering matrix was rigorously based in the known paper by R.Jost [2] (for further references in this direction see e.g. the books [3] and [4]).

Approaches to interpretation of solutions to the Schrödinger equation (so–called Gamow's vectors) corresponding to resonances are discussed in Refs. [5]–[7] (see also literature cited therein).

Idea to interpret resonances as poles of analytical continuation of the resolvent kernel for the Schrödinger operator (or matrix elements of the resolvent between suitable states) is realized in [8]–[16] (see also Refs. cited in these papers and in the books [17], [18]). Such interpretation became a basis for the perturbation theory for two–body resonances which is well developed now (see. [11], [12], [17]).

In the case when support of interaction in a system of two particles is compact with respect to relative coordinate, the approach [19] by P.Lax and R.Phillips may be applied (this approach was created initially for acoustical problems). The Lax–Phillips approach allows to describe resonances as a discrete spectrum of a dissipative operator representing generator of contracting semigroup. At present, the Lax–Phillips scheme is realized only in those scattering problems which generate the energy Riemann surface[2] with two sheets of the complex energy plane (see Refs. [20], [21]). In multichannel scattering problems, the approach above is partly realized in [22].

Beginning from 1970–es, the complex scaling method [10], [14] is applied to investigation of resonances (see also Refs. [15] and [18]). This method gives a possibility to rotate the continuous spectrum of Hamiltonians in such a way that certain sectors become accessible for observation on unphysical sheets neighboring with physical one. Resonances situated in these sectors turn into a part of the discrete spectrum of the Hamiltonian transformed. The complex scaling method may be applied in the cases when potentials are analytical functions of coordinates. This method allows to compute location of resonances in concrete physical problems (see, e.g. Refs. [15], [23]). As regards the structure of the scattering matrix and resolvent continued on unphysical sheets, this method gives not too large capacities.

Many important conceptual and constructive results (see [24]–[29]) for the physical sheet in three–body scattering problem are known to have been obtained on the base of the Faddeev equations [24] and their modifications. In particular the structure of resolvent and scattering operator was studied in details, completeness of the wave operators was proved and the coordinate asymptotics were studied in the case of quickly decreasing as well as Coulomb interactions[3] [24], [28], [29], [32]. Analogous results were obtained also for singular interactions

[2] The latter is understood usually as the Riemann surface of the resolvent kernel considered as a function of energy or as that of the resolvent bilinear form restricted on certain subsets of Hilbert space. Such operator–valued functions as the $T-$ and scattering matrices have usually the same Riemann surface since these functions are closely related to the resolvent.

[3] The new approaches [30], [31] (see also literature cited in [31]) have been developed recently in abstract scattering theory for N–body systems which allow to prove existence and asymptotical completeness of the wave operators in the case of pair interactions decreasing at the infinity as $r^{-\varrho}$, $\varrho > \sqrt{3}-1$, i.e. substantially

described by the boundary conditions of various types [32], [33]. On the base of the Faddeev equations, the methods of investigation of concrete physical systems were developed [29], [32], [34], [35].

As to the unphysical sheets, the situation is rather different. Here, when solving a concrete N–particle problem one usually restricts himself with developing some approximate numerical algorithm to search for resonances on unphysical sheets neighboring with physical one. A survey of different physical approaches to study of three-body resonances in the problems of nuclear physics can be found in Ref. [36]. A number of rigorous results (see [18]) is obtained in framework of the complex scaling method [10], [14], [15]. These results touch first of all the proofs of the existence of analytical continuation of resolvent in the N–body problem with potentials holomorphic with respect to the scale transforms. In Ref. [37], a proof is given for the existence of analytical continuation for the amplitudes of processes $2 \to 2$ in the N–particle system across the branches of continuous spectrum below the first breakup threshold of the system into three clusters.

A goal of the present work consists in analytical continuation and investigation of the structure of three–body T-matrix, scattering matrices and resolvent on unphysical sheets of the energy Riemann surface. The interaction potentials are supposed to be pairwise and decreasing in coordinate space not slower than exponentially. When constructing a theory of resonances in the two–body problem with such interactions one can use the coordinate as well as momentum representations. However, it is clear a priori that the analytical continuation of the three–body scattering theory equations [24], [29] on unphysical sheets becomes a very difficult problem if the equations are written in configuration space. Thing is that there exist noncompact (cylindrical) domains where interactions do not decrease. Meanwhile, the kernels of the integral equations continued increase exponentially. Their solutions have to increase exponentially, too. This means that the integral terms become divergent ones and the coordinate space equations lose a sense. In the momentum space, the integral terms of the scattering theory equations, e.g. the Faddeev equations for components of T–matrix, are actually the Cauchy type integrals analytical continuation of which (in a sense of distributions) is a solvable problem. A continuation of such kind on unphysical sheets neighboring with physical one was already realized for the s–wave Faddeev equations in the paper [38] (see also Ref. [36]) for the case of separable pair potentials. In the present paper, we construct a continuation of the Faddeev equations in the case of sufficiently arbitrary pair potentials not only on the neighboring unphysical sheets but also on all those remote sheets of the three–body Riemann surface where is possible to guide the spectral parameter (the energy z) going around two–body thresholds.

Main result of the paper consists in a basing of existence of analytical continuation on unphysical sheets of z for the Faddeev components $M_{\alpha\beta}(z)$, $\alpha, \beta = 1, 2, 3$, of the operator $T(z)$ and a construction of representations for them in terms of the physical sheet [see formula (6.8)]. According to the representations, the continued matrix $M(z)$ of the Faddeev components, $M = \{M_{\alpha\beta}\}$, is explicitly expressed on unphysical sheets in terms of this matrix itself taken on the physical one and some truncations of the scattering matrix. Kind of the truncation is determined by the index (number) of the unphysical sheet concerned. Note that structure of the representations is quite analogous to that of the representations found in the author's recent works [39] and [40] for analytical continuation of T–matrix in multichannel scattering problems with binary channels. Representations for analytical continuation of three–body scattering matrices follow immediately from the representation above for $M(z)$ [see Eqs. (6.9) and (6.11)]. As follows from the explicit representations (6.8), (6.9) and (6.11) obtained by us, the singularities of T–matrix, scattering matrices and resolvent on unphysical sheets differing from those on the physical one (poles at the discrete spectrum eigenvalues of the Hamiltonian), are actually singularities of the operator–valued functions of z inverse with respect to suitable truncations of the scattering matrix. Consequently, the resonances (i.e. the poles of T–matrix,

slower than Coulomb potentials.

scattering matrix and resolvent on unphysical sheets) are zeros of certain truncations of the scattering matrix taken on the physical sheet.

Results of the present paper were announced in the report [41].

The paper is organized as follows.

In Sec. 2, the main notations are described. Sec. 3 contains an information on analytical properties of the two–body $T-$ and scattering matrices which is necessary in subsequent sections. Sec. 4 is devoted to description of properties of the Faddeev components of three–body T–matrix and scattering matrices on the physical sheet of energy. In particular, the domains on the physical sheet are established where the half–on–shell Faddeev components and different truncations of the scattering matrices included in the representations (6.8), (6.9) and (6.11) may be considered as holomorphic functions. We justify these representations only on a certain part of the three–body Riemann surface which is described in Sec. 5. Analytical continuation of the Faddeev equations on unphysical sheets is described in Sec. 6. Also, in this section, the representations (6.8), (6.9) are (6.11) formulated for analytical continuation of the matrix $M(z)$, scattering matrices and resolvent respectively.

2. NOTATIONS

We consider a system of three spinless non–relativistic quantum particles. Movement of the mass center is assumed to be separated. For description of the system we use standard sets of the relative momenta k_α, p_α [29]. For example

$$\begin{aligned} k_1 &= \left[\frac{\mathrm{m}_2+\mathrm{m}_3}{2\mathrm{m}_2\mathrm{m}_3}\right]^{1/2}\cdot\frac{\mathrm{m}_2\mathrm{p}_3-\mathrm{m}_3\mathrm{p}_2}{\mathrm{m}_2+\mathrm{m}_3} \\ p_1 &= \left[\frac{\mathrm{m}_1+\mathrm{m}_2+\mathrm{m}_3}{2\mathrm{m}_1(\mathrm{m}_2+\mathrm{m}_3)}\right]^{1/2}\cdot\frac{(\mathrm{m}_2+\mathrm{m}_3)\mathrm{p}_1-\mathrm{m}_1(\mathrm{p}_2+\mathrm{p}_3)}{\mathrm{m}_1+\mathrm{m}_2+\mathrm{m}_3}, \end{aligned} \tag{2.1}$$

where m_α, p_α are masses and momenta of particles. Expressions for k_α, p_α with $\alpha = 2,3$ may be obtained from (2.1) by cyclic permutation of indices. Usually we combine relative momenta k_α, p_α into six-vectors $P = \{k_\alpha, p_\alpha\}$. A choice of certain pair $\{k_\alpha, p_\alpha\}$ fixes cartesian coordinate system in $\mathbf{R}^6$. Transition from one pair of momenta to another one means rotation in $\mathbf{R}^6$, $k_\alpha = c_{\alpha\beta}k_\beta + s_{\alpha\beta}p_\beta$, $p_\alpha = -s_{\alpha\beta}k_\beta + c_{\alpha\beta}p_\beta$, with coefficients $c_{\alpha\beta}$, $s_{\alpha\beta}$ depending on the particle masses only [29], such that $-1 < c_{\alpha\beta} < 0$, $s_{\alpha\beta}^2 = 1 - c_{\alpha\beta}^2$, $c_{\beta\alpha} = c_{\alpha\beta}$ and $s_{\beta\alpha} = -s_{\alpha\beta}$, $\beta \neq \alpha$.

In momentum representation, the Hamiltonian H of the three–body system under consideration is given by $(Hf)(P) = P^2 f(P) + \sum_{\alpha=1}^{3}(v_\alpha f)(P)$, $P^2 = k_\alpha^2 + p_\alpha^2$, $f \in \mathcal{H}_0 \equiv L_2(\mathbf{R}^6)$, with v_α, the pair potentials which are integral operators in k_α with kernels $v_\alpha(k_\alpha, k'_\alpha)$.

For the sake of definiteness all the potentials v_α, $\alpha = 1,2,3$, are supposed to be local. This means that the kernel of v_α depends on the difference of variables k_α and k'_α only, $v_\alpha(k_\alpha, k'_\alpha) = v_\alpha(k_\alpha - k'_\alpha)$. We consider two variants of the potentials v_α. In the first one, $v_\alpha(k)$ are holomorphic functions of the variable $k \in \mathbf{C}^3$ which satisfy the estimate

$$|v_\alpha(k)| \le \frac{c}{(1+|k|)^{\theta_0}}\mathrm{e}^{a_0|\mathrm{Im}k|} \quad \forall k \in \mathbf{C}^3 \tag{2.2}$$

with some $c > 0$, $a_0 > 0$ and $\theta_0 \in (3/2, 2)$. In the second variant, the potentials $v_\alpha(k)$ are holomorphic functions with respect to k in the strip $W_{2b} = \{k : k \in \mathbf{C}^3, |\mathrm{Im}k| < 2b\}$ only and obey for $k \in W_{2b}$ the condition (2.2) with $a_0 = 0$:

$$|v_\alpha(k)| \le \frac{c}{(1+|k|)^{\theta_0}} \quad \forall k : |\,\mathrm{Im}\,k| < 2b. \tag{2.3}$$

It is supposed that in both variants $v_\alpha(-k) = \overline{v_\alpha(k)}$. The latter condition guarantees self–adjointness of the Hamiltonian H on the set $\mathcal{D}(H) = \{f : \int(1+P^2)^2|f(P)|^2 dP < \infty\}$ [24].

Note that the first variant requirements of holomorphness of $v_\alpha(k)$ in all $\mathbf{C}^3$ and no more than exponential increasing (2.2) in $|\mathrm{Im}\, k|$ mean that these potentials have a compact support in the coordinate space. In the second variant, the potentials $v_\alpha(k)$ rewritten in the coordinate representation, decrease exponentially.

By h_α, $(h_\alpha f)(k_\alpha) = k_\alpha^2 f(k_\alpha) + (v_\alpha f)(k_\alpha)$, we denote the Hamiltonian of the pair subsystem α. The operator h_α acts in $L_2(\mathbf{R}^3)$. Due to conditions (2.2) and (2.3) its discrete spectrum $\sigma_d(h_\alpha)$ is negative and finite [18]. We enumerate the eigenvalues $\lambda_{\alpha,j} \in \sigma_d(h_\alpha)$, $\lambda_{\alpha,j} < 0$, $j = 1, 2, ..., n_\alpha$, $n_\alpha < \infty$, taking into account their multiplicity: number of times to meet an eigenvalue in the numeration equals to its multiplicity. Maximal of these numbers is denoted by $\lambda_{\max}$, $\lambda_{\max} = \max\limits_{\alpha,j} \lambda_{\alpha,j} < 0$. Notation $\psi_{\alpha,j}(k_\alpha)$ is used for respective eigenfunctions.

By $\sigma_d(H)$ and $\sigma_c(H)$ we denote respectively the discrete and continuous components of the spectrum $\sigma(H)$ of the Hamiltonian H. Note that $\sigma_c(H) = (\lambda_{\min}, +\infty)$ with $\lambda_{\min} = \min\limits_{\alpha,j} \lambda_{\alpha,j}$.

Notation H_0 is used for the operator of kinetic energy, $(H_0 f)(P) = P^2 f(P)$. $R_0(z)$ and $R(z)$ stand for the resolvents of the operators H_0 and $H:$ $R_0(z) = (H_0 - zI)^{-1}$ and $R(z) = (H - zI)^{-1}$ where in this case, I is the identity operator in $\mathcal{H}_0$.

Let $M_{\alpha\beta}(z) = \delta_{\alpha\beta} v_\alpha - v_\alpha R(z) v_\beta$, $\alpha, \beta = 1, 2, 3$, be the Faddeev components [24], [29] of the three–body T–matrix $T(z) = V - VR(z)V$ with $V = v_1 + v_1 + v_3$. Operators $M_{\alpha\beta}(z)$ satisfy the Faddeev equations [24], [29]

$$M_{\alpha\beta}(z) = \delta_{\alpha\beta} \mathbf{t}_\alpha(z) - \mathbf{t}_\alpha(z) R_0(z) \sum_{\gamma \neq \alpha} M_{\gamma\beta}(z), \quad \alpha = 1, 2, 3. \tag{2.4}$$

Here, the operator $\mathbf{t}_\alpha(z)$ has the kernel

$$\mathbf{t}_\alpha(P, P', z) = t_\alpha(k_\alpha, k'_\alpha, z - p_\alpha^2) \delta(p_\alpha - p'_\alpha), \tag{2.5}$$

where $t_\alpha(k, k', z)$ stands for the kernel of the pair T–matrix $t_\alpha(z) = v_\alpha - v_\alpha r_\alpha(z) v_\alpha$ with $r_\alpha(z) = (h_\alpha - z)^{-1}$.

It is convenient to rewrite the system (2.4) in the matrix form

$$M(z) = \mathbf{t}(z) - \mathbf{t}(z) \mathbf{R}_0(z) \Upsilon M(z), \tag{2.6}$$

with $\mathbf{t}(z) = \mathrm{diag}\{\mathbf{t}_1(z), \mathbf{t}_2(z), \mathbf{t}_3(z)\}$ and $\mathbf{R}_0(z) = \mathrm{diag}\{R_0(z), R_0(z), R_0(z)\}$. By Υ we denote a number 3×3–matrix with the elements $\Upsilon_{\alpha\beta} = 1 - \delta_{\alpha\beta}$. $M(z)$ is the operator matrix constructed of the components $M_{\alpha\beta}(z)$, $M = \{M_{\alpha\beta}\}$, $\alpha, \beta = 1, 2, 3$. The matrices M, $\mathbf{t}$, $\mathbf{R}_0$ and Υ are considered as operators in the Hilbert space $\mathcal{G}_0 = \mathop{\oplus}\limits_{\alpha=1}^{3} L_2(\mathbf{R}^6)$. By $\mathcal{Q}^{(k)}(z)$, $\mathcal{Q}^{(k)}(z) = (-\mathbf{t}(z)\mathbf{R}_0(z)\Upsilon)^k \mathbf{t}(z)$, we denote iterations of the absolute term $\mathcal{Q}^{(0)}(z) = \mathbf{t}(z)$ of (2.6).

The resolvent $R(z)$ is expressed in terms of the matrix $M(z)$ by formula [29]

$$R(z) = R_0(z) - R_0(z) \Omega M(z) \Omega^\dagger R_0(z), \tag{2.7}$$

where Ω, $\Omega : \mathcal{G}_0 \to \mathcal{H}_0$, stands for the matrix–row, $\Omega = (1,\ 1,\ 1)$. At the same time $\Omega^\dagger = \Omega^* = (1,\ 1,\ 1)^\dagger$. The symbol "$\dagger$" means transposition.

Throughout the paper we understand by $\sqrt{z - \lambda}$, $z \in \mathbf{C}$, $\lambda \in \mathbf{R}$, the main branch of the function $(z - \lambda)^{1/2}$. By $\hat{q}$ we denote usually the unit vector in the direction $q \in \mathbf{R}^N$, $\hat{q} = q/|q|$, and by S^{N-1} the unit sphere in $\mathbf{R}^N$, $\hat{q} \in S^{N-1}$. The inner product in $\mathbf{R}^N$ is denoted by $(\,\cdot\,,\,\cdot\,)$. Notation $\langle\,\cdot\,,\,\cdot\,\rangle$ is used for inner products in Hilbert spaces.

Let $\mathcal{H}^{(\alpha,j)} = L_2(\mathbf{R}^3)$ and $\mathcal{H}^{(\alpha)} = \mathop{\oplus}\limits_{j=1}^{n_\alpha} \mathcal{H}^{(\alpha,j)}$. By Ψ_α we denote operator acting from $\mathcal{H}^{(\alpha)}$ to $\mathcal{H}_0$ as $(\Psi_\alpha f)(P) = \sum\limits_{j=1}^{n_\alpha} \psi_{\alpha,j}(k_\alpha) f_j(p_\alpha)$, $f = (f_1, f_2, ..., f_{n_\alpha})^\dagger$. Notation Ψ_α^* is used for the operator

adjoint to Ψ_α. By Ψ we denote the block–diagonal matrix operator $\Psi = \mathrm{diag}\{\Psi_1, \Psi_2, \Psi_3\}$ which acts from $\mathcal{H}_1 = \mathop{\oplus}\limits_{\alpha=1}^{3} \mathcal{H}^{(\alpha)}$ to $\mathcal{G}_0$ and by Ψ^*, the operator adjoint to Ψ. Analogously to Ψ_α, Ψ^*_α, Ψ and Ψ^* we introduce the operators Φ_α, Φ^*_α, Φ and Φ^*, which are obtained from the former by replacement of the eigenfunctions $\psi_{\alpha,j}(k_\alpha)$ with the form–factors $\phi_{\alpha,j}(k_\alpha) = (v_\alpha\psi_{\alpha,j})(k_\alpha)$, $\alpha = 1, 2, 3$, $j = 1, 2, ..., n_\alpha$.

The pair T–matrix $t_\alpha(z)$ is known [24],[29] to be an analytical operator–valued function of the variable $z \in \mathbf{C}\backslash[0, +\infty)$ having at the points $z \in \sigma_d(h_\alpha)$ simple poles. Its kernel admits the representation

$$t_\alpha(k, k', z) = -\sum_{j=1}^{n_\alpha} \frac{\phi_{\alpha,j}(k)\overline{\phi_{\alpha,j}(k')}}{\lambda_{\alpha,j} - z} + \tilde{t}_\alpha(k, k', z), \tag{2.8}$$

where $\tilde{t}_\alpha(k, k'z)$ is a function holomorphic in $z \in \mathbf{C}\backslash[0, +\infty)$. Therefore

$$\mathbf{t}_\alpha(z) = -\Phi_\alpha \mathbf{g}_\alpha(z)\Phi^*_\alpha + \tilde{\mathbf{t}}_\alpha(z), \tag{2.9}$$

where the operator $\tilde{\mathbf{t}}_\alpha(z)$ has the kernel $\tilde{t}_\alpha(k_\alpha, k'_\alpha, z-p^2_\alpha)\delta(p_\alpha-p'_\alpha)$ and $\mathbf{g}_\alpha(z)$, $\mathbf{g}_\alpha(z)$=diag$\{g_{\alpha,1}(z)$, ..., $g_{\alpha,n_\alpha}(z)\}$, is the block–diagonal matrix with elements $g_{\alpha,j}(z)$, the operators in $\mathcal{H}^{(\alpha,j)}$ with singular kernels $g_{\alpha,j}(z)(p_\alpha, p'_\alpha, z) = \delta(p_\alpha - p'_\alpha)\,/(\lambda_{\alpha,j} - z + p^2_\alpha)$.

Below, we consider restrictions of different functions on the energy shell

$$k = \sqrt{z}\hat{k}, \quad \hat{k} \in S^2, \tag{2.10}$$

in the two–body problem and on the energy shells

$$P = \sqrt{z}\hat{P}, \quad \hat{P} \in S^5, \tag{2.11}$$

and

$$p_\alpha = \sqrt{z - \lambda_{\alpha,j}}\hat{p}_{\alpha,j}, \quad \hat{p}_{\alpha,j} \in S^2, \quad \alpha = 1, 2, 3, \quad j = 1, 2, ..., n_\alpha, \tag{2.12}$$

in the problem of three particles. In the last case the sets (2.11) and (2.12) are called respectively three–body and two–body energy shells.

Let $\mathcal{O}(\mathbf{C}^N)$ be the Fourier transform of the space $C_0^\infty(\mathbf{R}^N)$ (we deal with $N = 3$ or $N = 6$ only). Any $f(q) \in \mathcal{O}(\mathbf{C}^N)$ is a holomorphic function in variable $q = (q_1, q_2, ..., q_N) \in \mathbf{C}^N$ satisfying the estimates $\left|\frac{\partial^{|m|}}{\partial q_1^{m_1}\cdot...\cdot\partial q_N^{m_N}} f(q)\right| \leq c_\theta(f) \cdot \exp(a|\,\mathrm{Im}\, q|)(1 + |q|)^{-\theta}$, with a, the radius of the ball centered in the origin and containing the support of the Fourier pre–image of this function in $\mathbf{R}^N$, $|m| = m_1 + ... + m_N$, and $|\,\mathrm{Im}\, q| = \sqrt{\sum_{j=1}^N |\,\mathrm{Im}\, q_j|^2}$. As θ one can take arbitrary positive number. For fixed f and $m = (m_1, ..., m_N)$, the coefficient $c_\theta > 0$ depends only on θ.

Let $\mathrm{j}(z)$ be the operator restricting functions $f(k)$, $k \in \mathbf{R}^3$, on the energy shell (2.10) at $z = E \pm i0$, $E > 0$, and continuing them if possible, on the domain of complex values of the energy z. On the set $\mathcal{O}(\mathbf{C}^3)$ the operator $\mathrm{j}(z)$ acts as

$$\mathrm{j}(z)f(\hat{k}) = f(\sqrt{z}\hat{k}). \tag{2.13}$$

Its kernel is the holomorphic generalized function (distribution) [42] $\mathrm{j}(\hat{k}, k', z) = \delta(\sqrt{z}\hat{k} - k')$.

By $\mathrm{j}^\dagger(z)$ we denote the operator "transposed" with respect to $\mathrm{j}(z)$. Acting on $\varphi \in L_2(S^2)$ the operator $\mathrm{j}^\dagger(z)$ gives as a result the generalized functions (distributions) over $\mathcal{O}(\mathbf{C}^3)$,

$$(\mathrm{j}^\dagger(z)\varphi)(k) = \int_{S^2} d\hat{k}\; \delta(k - \sqrt{z}\hat{k}')\; \varphi(\hat{k}) = \frac{\delta(|k| - \sqrt{z})}{z}\varphi(\hat{k}), \tag{2.14}$$

i.e.

$$(\mathrm{j}^\dagger(z)\varphi, f) = \int\limits_{S^2} d\hat{k} f(\sqrt{z}\hat{k})\varphi(\hat{k}), \quad f \in \mathcal{O}(\mathbf{C}^3). \tag{2.15}$$

Remember that in terms of the operators $\mathrm{j}(z)$ and $\mathrm{j}^\dagger(z)$, the pair scattering matrices $s_\alpha(z)$, $s_\alpha(z) : L_2(S^2) \to L_2(S^2)$, look as (index of pair is omitted) [39]:

$$s(z) = \hat{I} + \mathrm{a}_0(z)\mathrm{j}(z)t(z)\mathrm{j}^\dagger(z), \tag{2.16}$$

where $\mathrm{a}_0(z) = -\pi i\sqrt{z}$ and $\hat{I}$ is the identity operator in $L_2(S^2)$. Analyticity domain of $s(z)$, $z \in \mathbf{C}$, is determined in a general way by properties of the pair potential v (see e.g., [3], [4], and also [39], [40]).

Let $\mathrm{J}_{\alpha,j}(z)$, $\alpha = 1,2,3$, $j = 1,2,...,n_\alpha$, be the operator of restriction on the energy shell (2.12). Its action on $\mathcal{O}(\mathbf{C}^3)$ is defined as

$$(\mathrm{J}_{\alpha,j}(z)f)(\hat{p}_\alpha) = f(\sqrt{z-\lambda_{\alpha,j}}\;\hat{p}_\alpha), \quad \alpha = 1,2,3, \quad j = 1,2,...,n_\alpha.$$

Operators $\mathrm{J}_{\alpha,j}(z)$ have the kernels $\mathrm{J}_{\alpha,j}(\hat{p}_\alpha, p'_\alpha, z) = \delta(\sqrt{z-\lambda_{\alpha,j}}\hat{p}_\alpha - p'_\alpha)$.

By $\mathrm{J}_0(z)$ we denote operator of restriction on the shell (2.11). On $\mathcal{O}(\mathbf{C}^6)$ this operator is defined as $(\mathrm{J}_0(z)f)(\hat{P}) = f(\sqrt{z}\hat{P})$. Its kernel is $\mathrm{J}_0(\hat{P}, P', z) = (\sqrt{z})^{-5}\delta(\sqrt{z}\hat{P} - P') = \delta(\sqrt{z} - |P'|)\delta(\hat{P}, \hat{P}')$.

Notations $\mathrm{J}^\dagger_{\alpha,j}(z)$ and $\mathrm{J}^\dagger_0(z)$ are used for respective "transposed" operators. Their action is defined similarly to (2.14),(2.15) as

$$(\mathrm{J}^\dagger_{\alpha,j}(z)\varphi)(p_\alpha) = \int\limits_{S^2} d\hat{p}'_\alpha \delta(p_\alpha - \sqrt{z-\lambda_{\alpha,j}}\hat{p}'_\alpha)\varphi(\hat{p}'_\alpha), \quad \varphi \in \hat{\mathcal{H}}^{(\alpha,j)},$$

$$(\mathrm{J}^\dagger_0(z)\varphi)(P) = \int\limits_{S^5} d\hat{P}'\delta(P - \sqrt{z}\hat{P}')\varphi(\hat{P}'), \quad \varphi \in \hat{\mathcal{H}}_0,$$

where $\hat{\mathcal{H}}^{(\alpha,j)} \equiv L_2(S^2)$ and $\hat{\mathcal{H}}_0 \equiv L_2(S^5)$. The generalized functions $\mathrm{J}^\dagger_{\alpha,j}(z)\varphi$ and $\mathrm{J}^\dagger_0(z)\varphi$ are elements of the spaces $\mathcal{O}'(\mathbf{C}^3)$ and $\mathcal{O}'(\mathbf{C}^6)$ of distributions over $\mathcal{O}(\mathbf{C}^3)$ and $\mathcal{O}(\mathbf{C}^6)$, respectively.

Operators $\mathrm{J}_{\alpha,j}$ and $\mathrm{J}^\dagger_{\alpha,j}$ are then combined into the block–diagonal matrices $\mathrm{J}^{(\alpha)}(z) = \mathrm{diag}\{\mathrm{J}_{\alpha,1}(z), ..., \mathrm{J}_{\alpha,n_\alpha}(z)\}$ and $\mathrm{J}^{(\alpha)\dagger}(z) = \mathrm{diag}\{\mathrm{J}^\dagger_{\alpha,1}(z), ..., \mathrm{J}^\dagger_{\alpha,n_\alpha}(z)\}$. Latter are used to construct operators $\mathrm{J}_1(z) = \mathrm{diag}\{\mathrm{J}^{(1)}(z), \mathrm{J}^{(2)}(z), \mathrm{J}^{(3)}(z)\}$ and $\mathrm{J}^\dagger_1(z) = \mathrm{diag}\{\mathrm{J}^{(1)\dagger}(z), \mathrm{J}^{(2)\dagger}(z), \mathrm{J}^{(3)\dagger}(z)\}$. The action of $\mathrm{J}^{(\alpha)}(z)$ and $\mathrm{J}_1(z)$ on elements of the spaces respectively, $\mathcal{O}^{(\alpha)} = \mathop{\times}\limits_{\alpha=1}^{n_\alpha} \mathcal{O}^{(\alpha,j)}$, $\mathcal{O}^{(\alpha,j)} \equiv \mathcal{O}(\mathbf{C}^3)$ and $\mathcal{O}_1 = \mathop{\times}\limits_{\alpha=1}^{3} \mathcal{O}^{(\alpha)}$ can be understood by the definition of the operators $\mathrm{J}_{\alpha,j}(z)$. The operators $\mathrm{J}^{(\alpha)\dagger}(z)$ act from $\hat{\mathcal{H}}^{(\alpha)} \equiv \mathop{\oplus}\limits_{j=1}^{n_\alpha} \hat{\mathcal{H}}^{(\alpha,j)}$ to the space of analytical distributions $\mathcal{O}^{(\alpha)\prime}$ over $\mathcal{O}^{(\alpha)}$. In its turn the operator $\mathrm{J}^\dagger_1(z)$ acts from $\hat{\mathcal{H}}_1 = \mathop{\oplus}\limits_{\alpha=1}^{3} \hat{\mathcal{H}}^{(\alpha,j)}$ to the space of analytical distributions $\mathcal{O}'_1$ over $\mathcal{O}_1$.

At last, we use the block–diagonal operator 3×3–matrices $\mathbf{J}_0(z) = \mathrm{diag}\{\mathrm{J}_0(z), \mathrm{J}_0(z), \mathrm{J}_0(z)\}$ and $\mathbf{J}^\dagger_0(z) = \mathrm{diag}\{\mathrm{J}^\dagger_0(z), \mathrm{J}^\dagger_0(z), \mathrm{J}^\dagger_0(z)\}$, constructed of the operators $\mathrm{J}_0(z)$ and $\mathrm{J}^\dagger_0(z)$, respectively as well as the operators $\mathbf{J}(z) = \mathrm{diag}\{\mathrm{J}_0(z), \mathrm{J}_1(z)\}$ $\mathbf{J}^\dagger(z) = \mathrm{diag}\{\mathrm{J}^\dagger_0(z), \mathrm{J}^\dagger_1(z)\}$. Action of these operators is clear due to definitions of the operators J_0, J_1, $\mathrm{J}^\dagger_0$ and $\mathrm{J}^\dagger_1$. In particular the operator $\mathbf{J}^\dagger(z)$ acts from the space $\hat{\mathcal{G}}_0 = \mathop{\oplus}\limits_{\alpha=1}^{3} \hat{\mathcal{H}}_0$ to the space $\mathop{\times}\limits_{\alpha=1}^{3} \mathcal{O}'(\mathbf{C}^6)$.

The identity operators in the spaces $\hat{\mathcal{H}}_0$, $\hat{\mathcal{G}}_0$, $\hat{\mathcal{H}}_1$ and $\hat{\mathcal{H}}_0 \oplus \hat{\mathcal{H}}_1$ are denoted by $\hat{I}_0$, $\hat{\mathbf{I}}_0$, $\hat{I}_1$ and $\hat{\mathbf{I}}$ respectively.

3. ANALYTICAL CONTINUATION OF THE T– AND SCATTERING MATRICES IN THE TWO–BODY PROBLEM

In this section we remember some analytical properties of the pair T–matrices which will be necessary further when posing the three–body problem. Note that above properties are well known (see e.g., Refs [4], [3] and also [36]) for a wide class of the potentials $v_\alpha(x)$. As a matter of fact we want to expose here only an explicit representation for the two–body T–matrix on unphysical sheet which is a particular case of the explicit representations constructed in the author's work [39] (see Theorem 2 in [39] and comments to it) for a rather more general situation of analytical continuation of T–matrix on unphysical sheets in the multichannel problem with binary channels.

Throughout the section we shall consider a fixed pair subsystem. Therefore its index will be omitted in notations. Statements will be given for the first variant of the potentials (2.2). If it will be necessary, different assertions for the second variant (2.3) will be written in brackets. Also, we use the notation

$$\mathcal{P}_b = \left\{ z: \quad \mathrm{Re}\, z > -b^2 + \frac{1}{4b^2}(\mathrm{Im}\, z)^2 \right\}. \tag{3.1}$$

Remember that the energy Riemann surface in the two–body problem coincides with that of the function $z^{1/2}$. On the physical sheet, $z^{1/2} = \sqrt{z}$, and on the unphysical one, $z^{1/2} = -\sqrt{z}$. For these sheets we use the notations respectively, Π_0 and Π_1.

Representation for continuation of $t(z)$ on unphysical sheet which will be used further, is described by the following statement which is one–channel variant of Theorem 2 of Ref. [39].

Theorem 1. *The two–body T–matrix $t(z)$ allows analytical continuation in variable z on the sheet Π_1 (on the domain $\mathcal{P}_b \bigcap \Pi_1$) as a bounded operator in $L_2(\mathbf{R}^3)$. Result of the continuation $t(z)|_{\Pi_1}$ $\left(t(z)|_{\mathcal{P}_b \bigcap \Pi_1} \right)$ is expressed by T– and S–matrices on the physical sheet:*

$$t(z)\big|_{\Pi_1} = t(z) - \mathrm{a}_0(z)\, \tau(z) \tag{3.2}$$

where $\tau(z) = (t\mathrm{j}^\dagger s^{-1} \mathrm{j} t)(z)$. The kernel $t(k,k',z)|_{\Pi_1}$ is a holomorphic function of variables $k, k' \in \mathbf{C}^3$ and $z \in \Pi_1 \setminus (\sigma_{\mathrm{res}} \bigcup \sigma_d(h))$ $\left(k, k' \in W_b \text{ and } z \in \mathcal{P}_b \bigcap \Pi_1 \setminus (\sigma_{\mathrm{res}} \bigcup \sigma_d(h)) \right)$. Here, σ_{res} is a set of the points $z \in \mathbf{C} \setminus \overline{\sigma(h)}$ $\left(z \in \mathcal{P}_b \setminus \overline{\sigma(h)} \right)$ where the operator $[s(z)]^{-1}$ does not exist.

Emphasize that for the second variant of potentials (2.3), the existence of the continuation of $t(z)$ on unphysical sheet is guaranteed by Theorem 1 for the domain $\mathcal{P}_b \bigcap \Pi_1$ bounded by the parabola $\mathrm{Im}\sqrt{z} = b$, inside of which the function $v\left(\sqrt{z}(\hat{k} - \hat{k}')\right)$ is holomorphic in z for arbitrary $\hat{k}, \hat{k}' \in S^2$. Note also that the operator $(\mathrm{j}t\mathrm{j}^\dagger)(z)$, included in Eq. (2.16), is a compact operator in $C(S^2)$ [39]. Consequently on the domain of its analyticity $\Pi_0 \setminus \overline{\sigma(h)}$ $(\mathcal{P}_b \bigcap \Pi_0 \setminus \overline{\sigma(h)})$ on the physical sheet, one can apply to the equation

$$s(z)\mathcal{A} = 0 \tag{3.3}$$

the Fredholm alternative [18] (see Ref. [39]). This means that the set σ_{res} being countable, has not concentration points in $\mathbf{C} \setminus \overline{\sigma(h)}$ $\left(\mathcal{P}_b \setminus \overline{\sigma(h)} \right)$.

On the physical sheet Π_0, the pair T–matrix admits the representation (2.8). It follows from the Lippmann–Schwinger equation for ϕ_j, $j = 1, 2, ..., n$,

$$\phi_j(k) = -\int\limits_{\mathbf{R}^3} dq\; v(k,q) \frac{1}{q^2 - \lambda_j} \phi_j(q), \quad \lambda_j < 0, \tag{3.4}$$

that form–factor $\phi_j(k)$ admits analytical continuation in k on $\mathbf{C}^3$ (on W_{2b}) and at the same time, it satisfies the type (2.2) estimate where one has to replace θ_0 with a number θ, $1 < \theta < \theta_0$, which can be taken in any close vicinity of θ_0 [24]. Hence the eigenfunction

$$\psi_j(k) = -\frac{\phi_j(k)}{k^2 - \lambda_j} \tag{3.5}$$

of h admits also an analytical continuation on $\mathbf{C}^3$ (on W_{2b}) with the exception of the set $\{k \in \mathbf{C}^3 : k^2 = \lambda_j\}$ where $\psi_j(k)$ has singularities (turning for $k = \sqrt{z}\hat{k}$, $\hat{k} \in S^2$, into a pole in energy z at $z = \lambda_j$).

The regular summand $\tilde{t}(k, k'z)$ of the kernel of $t(z)$ is holomorphic function in variables $k, k' \in \mathbf{C}^3$, $z \in \Pi_0$ $\quad (k, k' \in W_b,\ z \in \mathcal{P}_b \bigcap \Pi_0)$ and admits the estimate

$$|\tilde{t}(k, k'z)| < c(1 + |k - k'|)^{-\theta} \cdot \exp[a(|\operatorname{Im} k| + |\operatorname{Im} k'|)],$$

with arbitrary $\theta \in (1, \theta_0)$.

As to continuation of $t(z)\big|_{\Pi_1}$, it follows from Eq. (3.2) that the points $z \in \sigma_d(h)$ give to it generally speaking, poles of the first order. One can easily check however that if eigenvalue $\lambda \in \sigma_d(h)$ is simple then the respective singularities of the both summands of (3.2) compensate each other and the pole of $t(z)\big|_{\Pi_1}$ does not appear at $z = \lambda$. It follows from the Fredholm analytical alternative [18] for Eq. (3.3) only that poles of $t(z)\big|_{\Pi_1}$ at $z \in \sigma_{\rm res}$ are of a finite order and no more. It is easily to show that if $\mathcal{A}(\hat{k})$ is a nontrivial solution of Eq. (3.3) at $z \in \sigma_{\rm res}$, $z \notin \sigma_d(h)$, then the Schrödinger equation has at this z a nontrivial (resonance) solution. Asymptotics of this solution $\psi^{\#}_{\rm res}(x)$ in configuration space, $x \in \mathbf{R}^3$, is exponentially increasing: $\psi^{\#}_{\rm res}(x) \underset{x\to\infty}{=} (\mathcal{A}(-\hat{x}) + o(1))\,\dfrac{\mathrm{e}^{-i\sqrt{z}|x|}}{|x|}$. The function $\psi^{\#}_{\rm res}(x)$ is so–called Gamow vector corresponding to resonance at the energy z (see e.g., Refs. [3], [6], [7]). The function $\mathcal{A}(\hat{k})$ makes a sense to the breakup amplitude of the resonance state[4].

The formula for analytical continuation of the scattering matrix on unphysical sheet Π_1 (on the set $\mathcal{P}_b \bigcap \Pi_1$) follows immediately from Eq. (3.2) (see Ref. [39]),

$$s(z)\big|_{\Pi_1} = \mathcal{E}[s(z)]^{-1}\mathcal{E}, \tag{3.6}$$

where $\mathcal{E}$ stands for the inversion in $L_2(S^2)$, $(\mathcal{E}f)(\hat{k}) = f(-\hat{k})$.

Utilizing (3.2) one can easily to get the explicit representation in terms of the physical sheet as well for analytical continuation on Π_1 (on $\mathcal{P}_b \bigcap \Pi_1$) of the resolvent $r(z)$ kernel[5]:

$$r(z)\big|_{\Pi_l} = r + \mathrm{a}_0(I - rv)\mathrm{j}^{\dagger}s^{-1}\mathrm{j}(I - vr). \tag{3.7}$$

The continuation has to be understood in a sense of generalized functions (distributions) over $\mathcal{O}(\mathbf{C}^3)$: one has to continue the bilinear form $\Phi(z) = (r(z)f_1, f_2) \equiv \int_{\mathbf{R}^3} dk \int_{\mathbf{R}^3} dk'\, f_2(k)\, r(k, k', z) f_1(k')$

[4] Analogous assertion takes place as well in the multichannel scattering problem with m binary channels: solution $\mathcal{A} = (\mathcal{A}_1, \mathcal{A}_2, ..., \mathcal{A}_m)$ to the equation $s_l(z)\mathcal{A} = 0$ at resonance energy $z \in \sigma^l_{\rm res}$ (in notations of Ref. [39]) represents amplitudes (i.e. coefficients at spherical waves in coordinate asymptotics of the channel components of solution to respective Schrödinger equation) $\mathcal{A}_1(\hat{k}_1), \mathcal{A}_2(\hat{k}_2), ..., \mathcal{A}_m(\hat{k}_m)$ of resonance on the sheet Π_l to breakup into channels 1,2,...,m, respectively.

[5] Similar representations take place as well in the case of the multichannel problem. In notations of Ref. [39] read them as $r(z)\big|_{\Pi_l} = r + (I - rv)\mathrm{J}^{\dagger}ALs_l^{-1}\mathrm{J}(I - vr)$.

4. MATRIX $M(z)$ AND THREE–BODY SCATTERING MATRICES ON THE PHYSICAL SHEET

At the beginning, remember shortly principal properties [24], [29] of the Faddeev equations (2.6) for the matrix $M(z)$ and properties of the kernels $M_{\alpha\beta}(P,P',z)$ at real arguments $P,P'\in\mathbf{R}^6$. To formulate these properties we cite here the following definition [24].

The operator–valued function $\mathcal{Q}_{\alpha\beta}(z)$ of variable $z\in\mathbf{C}$, $\mathcal{Q}_{\alpha\beta}(z):\ \mathcal{H}_0\to\mathcal{H}_0$, is the type $\mathcal{D}_{\alpha\beta}$ function, $\alpha,\beta=1,2,3$, if it admits the representation

$$\begin{aligned}\mathcal{Q}_{\alpha\beta}(z)=\mathcal{F}_{\alpha\beta}(z)+\Phi_\alpha\mathbf{g}_\alpha(z)\mathcal{I}_{\alpha\beta}(z)+\\ +\mathcal{J}_{\alpha\beta}(z)\mathbf{g}_\beta(z)\Phi^*_\beta+\Phi_\alpha\mathbf{g}_\alpha(z)\mathcal{K}_{\alpha\beta}(z)\mathbf{g}_\beta(z)\Phi^*_\beta.\end{aligned}\tag{4.1}$$

The operator–valued functions $\mathcal{F}_{\alpha\beta}(z):\ \mathcal{H}_0\to\mathcal{H}_0$, $\mathcal{I}_{\alpha\beta}(z):\ \mathcal{H}_0\to\mathcal{H}^{(\alpha)}$, $\mathcal{J}_{\alpha\beta}(z):\ \mathcal{H}^{(\beta)}\to\mathcal{H}_0$ and $\mathcal{K}_{\alpha\beta}(z):\ \mathcal{H}^{(\beta)}\to\mathcal{H}^{(\alpha)}$ are called components of the function $\mathcal{Q}_{\alpha\beta}(z)$. If $\mathcal{Q}_{\alpha\beta}(z)$ is an integral operator then its kernel is called kernel of the type $\mathcal{D}_{\alpha\beta}$.

Let $\mathcal{N}(P,\theta)=\sum\limits_{\alpha,\beta,\ \alpha\neq\beta}(1+|p_\alpha|)^{-\theta}(1+|p_\beta|)^{-\theta}$. A function $\mathcal{Q}(z)$ of the type $\mathcal{D}_{\alpha\beta}$ is called the class $\mathcal{D}_{\alpha\beta}(\theta,\mu)$ function if its components $\mathcal{F}_{\alpha\beta}$, $\mathcal{I}_{\alpha\beta}$, $\mathcal{J}_{\alpha\beta}$ and $\mathcal{K}_{\alpha\beta}$ are integral operators and for the kernels $\mathcal{F}_{\alpha\beta}(P,P',z)$ at P, P', ΔP, $\Delta P'\in\mathbf{R}^6$, the estimates

$$|\mathcal{F}_{\alpha\beta}(P,P',z)|\leq c\,\mathcal{N}(P,\theta)(1+{p'_\beta}^2)^{-1},\tag{4.2}$$

$$\begin{aligned}|\mathcal{F}_{\alpha\beta}(P+\Delta P,P'+\Delta P',z+\Delta z)-\mathcal{F}(P,P',z)|\leq\\ \leq c\,\mathcal{N}(P,\theta)(1+{p'_\beta}^2)^{-1}(|\Delta P|^\mu+|\Delta P'|^\mu+|\Delta z|^\mu)\end{aligned}\tag{4.3}$$

with certain $c>0$ take place and at the same time, the kernels $\mathcal{I}_{\alpha,j;\beta}(p_\alpha,P',z)$, $\mathcal{J}_{\alpha;\beta,k}(P,p'_\beta,z)$ and $\mathcal{K}_{\alpha,j;\beta,k}(p_\alpha,p'_\beta,z)$ satisfy inequalities which may be got from (4.2) and (4.3) if to take respectively, $k_\alpha=0$, $k'_\beta=0$ or simultaneously, $k_\alpha=0$, $k'_\beta=0$.

Let $\mathcal{Q}^{(n)}(z)$ be an iteration of the absolute term of Eq. (2.6). In a contrast to $\mathcal{Q}^{(0)}(z)=\mathbf{t}(z)$ kernels of the operators $\mathcal{Q}^{(n)}(z)$ at $n>0$ do not include δ–functions. Moreover, it follows from the representation (2.9) for $\mathbf{t}_\alpha(z)$ explicitly manifesting a contribution of the discrete spectrum of pair subsystems, that matrix elements $\mathcal{Q}^{(n)}_{\alpha\beta}(z)$, $\alpha,\beta=1,2,3$, of the operators $\mathcal{Q}^{(n)}(z)$ with $n\geq 1$ are actually functions of the $\mathcal{D}_{\alpha\beta}$ type. Their components $\mathcal{F}^{(n)}_{\alpha\beta}(z)$, $\mathcal{I}^{(n)}_{\alpha\beta}(z)$, $\mathcal{J}^{(n)}_{\alpha\beta}(z)$ and $\mathcal{K}^{(n)}_{\alpha\beta}(z)$ at $z\in\mathbf{C}\setminus[\lambda_{\min},+\infty)$ are bounded operators depending on z analytically. In the case of potentials (2.2) and (2.3), the Hölder index of smoothness μ for their kernels with respect to variables P,P',p_α and p'_β at $z\notin[\lambda_{\min},+\infty)$ equals to 1. If $n\leq 3$ then as $\operatorname{Im}z\to 0$, $\operatorname{Re}z\in[\lambda_{\min},+\infty)$ the kernels $\mathcal{F}^{(n)}_{\alpha\beta}$ $\mathcal{I}^{(n)}_{\alpha,j;\beta}$, $\mathcal{J}^{(n)}_{\alpha;\beta,k}$, and $\mathcal{K}^{(n)}_{\alpha,j;\beta,k}$ have so–called *minor* (three–particle) singularities (see Refs. [24] and [29]) weakening with growing n. At $n\geq 4$ such singularities do not appear at all and these kernels become Hölder functions in all their variables including the limit values $z=E\pm i0$, $E\in(\lambda_{\min},+\infty)$. More precise statement [24] is following: the operator–valued functions $\mathcal{Q}^{(n)}_{\alpha\beta}(z)$ at $n\geq 4$ belong to the type $\mathcal{D}_{\alpha\beta}(\theta,\mu)$, $0<\theta<\theta_0$, $0<\mu<\frac{1}{8}$, uniformly with respect to z changing on arbitrary bounded set in the complex plane $\mathbf{C}$ with cut along the ray $[\lambda_{\min},+\infty)$. One can take as θ, $\theta<\theta_0$, any number as close as possible to θ_0. Thus, instead of $M(z)$ it is convenient [24] to come to the new unknown $\mathcal{W}(z)=M(z)-\sum_{n=0}^{3}\mathcal{Q}^{(n)}(z)$, satisfying the equation

$$\mathcal{W}(z)=\mathcal{W}^{(0)}(z)-\mathbf{t}(z)\mathbf{R}_0(z)\Upsilon\mathcal{W}(z)\tag{4.4}$$

analogous to Eq. (2.6) but with another absolute term $\mathcal{W}^{(0)}(z)=\mathcal{Q}^{(4)}(z)$.

Immersion of Eq. (4.4) in the Banach space $\mathcal{B}(\theta,\mu)$ (a description of the latter see in Refs. [24], or [29]) leads one to the following important

Theorem 2 (L.D.Faddeev [24]). *Eq. (2.6) is uniquely solvable at $z \notin \overline{\sigma_d(H)}$. Its solution $M(z)$ admits the representation*

$$M(z) = \sum_{n=0}^{3} \mathcal{Q}^{(n)}(z) + \mathcal{W}(z), \tag{4.5}$$

where the operator–valued function $\mathcal{W}(z)$ is holomorphic in variable z at $z \notin \overline{\sigma(H)}$ and its components $\mathcal{W}_{\alpha\beta}(z)$ belong to the classes $\mathcal{D}_{\alpha\beta}(\theta,\mu)$, $3/2 < \theta < \theta_0$, $0 < \mu < \frac{1}{8}$, uniformly with respect to z changing in arbitrary bounded set of the complex plane $\mathbf{C}$ with cut along the ray $[\lambda_{\min}, +\infty)$ and removed neighborhoods of the points of $\sigma_d(H)$.

Remember now structure of the scattering operator $\mathbf{S}$ [24], [29] for the system of three particles. For this purpose we introduce the operator–valued function $\mathcal{T}(z)$, $\mathcal{T}(z) : \mathcal{H}_0 \oplus \mathcal{H}_1 \to \mathcal{H}_0 \oplus \mathcal{H}_1$, of $z \in \mathbf{C} \setminus \overline{\sigma(H)}$,

$$\mathcal{T}(z) \equiv \begin{pmatrix} \Omega M(z)\Omega^\dagger & \Omega M(z)\Upsilon\Psi \\ \Psi^*\Upsilon M(z)\Omega^\dagger & \Psi^*(\Upsilon\mathbf{v} + \Upsilon M(z)\Upsilon)\Psi \end{pmatrix}, \tag{4.6}$$

with $\mathbf{v} = \mathrm{diag}\{v_1, v_2, v_3\}$. Note that $\mathcal{T}_{00}(z) = \Omega M(z)\Omega^\dagger \equiv T(z)$, $\mathcal{T}_{00}(z) : \mathcal{H}_0 \to \mathcal{H}_0$. The rest of the components $\mathcal{T}_{01}(z) : \mathcal{H}_1 \to \mathcal{H}_0$, $\mathcal{T}_{10}(z) : \mathcal{H}_0 \to \mathcal{H}_1$ and $\mathcal{T}_{11}(z) : \mathcal{H}_1 \to \mathcal{H}_1$ is expressed by the transition operators [29] (see also [34]) $U_0(z) = \Omega M(z)\Upsilon$, $U_0^\dagger = \Upsilon M(z)\Omega^\dagger$ and $U(z) = \Upsilon\mathbf{v} + \Upsilon M(z)\Upsilon$: $\mathcal{T}_{01} = U_0\Psi$, $\mathcal{T}_{10} = \Psi^* U_0^\dagger$ and $\mathcal{T}_{11} = \Psi^* U \Psi$. The operator $\mathcal{T}(z)$ is a matrix integral operator with kernels $\mathcal{T}_{00}(P, P', z)$, $\mathcal{T}_{\alpha,i;\,0}(p_\alpha, P', z)$, $\mathcal{T}_{0;\,\beta,j}(P, p'_\beta, z)$ and $\mathcal{T}_{\alpha,i;\,\beta,j}(p_\alpha, p'_\beta, z)$, $\alpha = 1,2,3$, $i = 1,2,...,n_\alpha$, $\beta = 1,2,3$, $j = 1,2,...,n_\beta$, properties of which are determined including the limit points $z = E \pm i0$, $E > \lambda_{\min}$, by Theorem 2.

By $\hat{\mathcal{T}}(z)$, $\hat{\mathcal{T}}(z) : \hat{\mathcal{H}}_0 \oplus \hat{\mathcal{H}}_1 \to \hat{\mathcal{H}}_0 \oplus \hat{\mathcal{H}}_1$, we denote analytical continuation in $\mathbf{C}^\pm$ (see Theorems 3, 6 and 7) of the operators $\hat{\mathcal{T}}(E \pm i0)$ having the kernels

$$\begin{aligned}
(\hat{\mathcal{T}}(E \pm i0))_{00}(\hat{P}, \hat{P}') &= \mathcal{T}_{00}(\pm\sqrt{E}\hat{P}, \pm\sqrt{E}\hat{P}', E \pm i0), \quad E > 0; \\
(\hat{\mathcal{T}}(E \pm i0))_{0;\,\beta,j}(\hat{P}, \hat{p}'_\beta) &= \mathcal{T}_{0;\,\beta,j}(\pm\sqrt{E}\hat{P}, \pm\sqrt{E-\lambda_{\beta,j}}\hat{p}'_\beta, E \pm i0), \quad E > 0; \\
(\hat{\mathcal{T}}(E \pm i0))_{\alpha,i;0}(\hat{p}_\alpha, \hat{P}') &= \mathcal{T}_{\alpha,i;0}(\pm\sqrt{E-\lambda_{\alpha,i}}\hat{p}_\alpha, \pm\sqrt{E}\hat{P}', E \pm i0), \quad E > 0; \\
(\hat{\mathcal{T}}(E \pm i0))_{\alpha,i;\,\beta,j}(\hat{p}_\alpha, \hat{p}'_\beta) &= \mathcal{T}_{\alpha,i;\,\beta,j}(\pm\sqrt{E-\lambda_{\alpha,i}}\hat{p}_\alpha, \pm\sqrt{E-\lambda_{\beta,j}}\hat{p}'_\beta, E \pm i0), \\
&\qquad\qquad E > \max\{\lambda_{\alpha,i},\, \lambda_{\beta,j}\}.
\end{aligned}$$

We assume by definition that the product $(\mathbf{J}\mathcal{T}\mathbf{J}^\dagger)(z)$ coincides with $\hat{\mathcal{T}}(z)$,

$$(\mathbf{J}\mathcal{T}\mathbf{J}^\dagger)(z) = \begin{pmatrix} (\mathrm{J}_0\mathcal{T}_{00}\mathrm{J}_0^\dagger)(z) & (\mathrm{J}_0\mathcal{T}_{01}\mathrm{J}_1^\dagger)(z) \\ (\mathrm{J}_1\mathcal{T}_{10}\mathrm{J}_0^\dagger)(z) & (\mathrm{J}_1\mathcal{T}_{11}\mathrm{J}_1^\dagger)(z) \end{pmatrix} \equiv \hat{\mathcal{T}}(z). \tag{4.7}$$

Elements of the matrix $(\mathbf{J}\mathcal{T}\mathbf{J}^\dagger)(z)$ are expressed in terms of amplitudes of different processes taking place in the three–body system under consideration [29] (see also Sec. 7 of [43]).

The scattering operator $\mathbf{S}$ is unitary one in the space $\mathcal{H}_0 \oplus \mathcal{H}_1$ and as well as $\mathcal{T}$, it has a natural block structure. Its components $\mathbf{S}_{00}$, $\mathbf{S}_{0;\beta,j}$, $\mathbf{S}_{\alpha,i;0}$, $\mathbf{S}_{\alpha,i;\beta,j}$ have the kernels, respectively

$$\mathbf{S}_{00}(P, P') = \delta(P - P') - 2\pi i\,\delta(P^2 - P'^2)\mathcal{T}_{00}(P, P', P'^2 + i0), \tag{4.8}$$

$$\mathbf{S}_{0;\beta,j}(P, p'_\beta) = -2\pi i\,\delta(P^2 - p'^2_\beta - \lambda_{\beta,j})\mathcal{T}_{0;\beta,j}(P, p'_\beta, \lambda_{\beta,j} + p'^2_\beta + i0), \tag{4.9}$$

$$\mathbf{S}_{\alpha,i;0}(p_\alpha, P') = -2\pi i\,\delta(\lambda_{\alpha,i} + p^2_\alpha - P'^2)\mathcal{T}_{\alpha,i;0}(p_\alpha, P', P'^2 + i0), \tag{4.10}$$

$$\begin{aligned}
\mathbf{S}_{\alpha,i;\beta,j}(p_\alpha, p'_\beta) &= \delta_{\alpha\beta}\delta_{ij}\delta(p_\alpha - p'_\beta) - \\
&\quad -2\pi i\,\delta(\lambda_{\alpha,i} + p^2_\alpha - \lambda_{\beta,j} - p^2_\beta)\mathcal{T}_{\alpha,i;\beta,j}(p_\alpha, p'_\beta, \lambda_{\beta,j} + p^2_\beta + i0).
\end{aligned} \tag{4.11}$$

Scattering matrices arise from $\mathbf{S}$ in the spectral decomposition for H as operators acting in the "cross section" (at fixed energy) of the space $\mathcal{H}_0 \oplus \mathcal{H}_1$ in the Neumann direct integral [28]. Extraction of the scattering matrix from $\mathbf{S}$ is related as a matter of fact to the replacements $|P|^2 \to E$, $\lambda_{\alpha,i} + p_\alpha^2 \to E$, $\alpha = 1,2,3$, $i = 1,2,...,n_\alpha$, in expressions (4.8)—(4.11) and then to the factorization of dependence of the kernels of $\mathbf{S}$ on the energies E and E',

$$\mathbf{S}(E,E') = -\pi i \delta(E-E')\tilde{\vartheta}(E) S'(E+i0)\tilde{\vartheta}(E'), \tag{4.12}$$

where $\tilde{\vartheta}(E)$ is a diagonal matrix–function constructed of the Heaviside functions $\vartheta(E)$ and $\vartheta(E-\lambda_{\beta,j})$: $\tilde{\vartheta}(E) = \mathrm{diag}\{\vartheta(E), \vartheta(E-\lambda_{1,1}), ... \,\vartheta(E-\lambda_{1,n_1}), \vartheta(E-\lambda_{2,1}), ... \,\vartheta(E-\lambda_{2,n_2}), \vartheta(E-\lambda_{3,1}), ... \,\vartheta(E-\lambda_{3,n_3})\}$. At $z \in \mathbf{C}$ we understand by $S'(z)$ the operator–valued function $S'(z) = A^{-1}(z)\,\hat{\mathbf{I}} + \hat{\mathcal{T}}(z)$. Here and all over further, $A(z) = \mathrm{diag}\{A_0(z),\, A_1(z)\}$ with $A_0(z) = -\pi i z^2$ and $A_1(z) = \mathrm{diag}\{A^{(1)},\, A^{(2)},\, A^{(3)}\}$ where in its turn, $A^{(\alpha)}(z) = \mathrm{diag}\{A_{\alpha,1}(z), ..., A_{\alpha,n_\alpha}(z)\}$ with $A_{\alpha,j}(z) = -\pi i \sqrt{z - \lambda_{\alpha,j}}$.

Continuing the factorization, $S'(z) = S(z)A^{-1}(z) = A^{-1}(z)S^\dagger(z)$, corresponding to separating in (4.12) the multiplier $-\pi i A^{-1}(E+i0)$ as a derivative of measure in the Neumann integral above [28] for $\mathcal{H}_0 \oplus \mathcal{H}_1$, one comes to the scattering matrices

$$S(z) = \hat{\mathbf{I}} + (\mathbf{J}\mathcal{T}\mathbf{J}^\dagger\, A)(z) \;\text{ and }\; S^\dagger(z) = \hat{\mathbf{I}} + (A\,\mathbf{J}\mathcal{T}\mathbf{J}^\dagger)(z). \tag{4.13}$$

In a contrast to Ref. [28] it is more convenient for us to use namely this, nonsymmetrical, form of the scattering matrices. Matrices $S(z)$ and $S^\dagger(z)$ are considered as operators in $\hat{\mathcal{H}}_0 \oplus \hat{\mathcal{H}}_1$. At $z = E + i0$, $E > 0$, these operators are unitary. At $z = E + i0$, $E < 0$, there are certain truncations of $S(z)$ and $S^\dagger(z)$ determined by the number of open channels which are unitary in $\hat{\mathcal{H}}_0 \oplus \hat{\mathcal{H}}_1$, namely the matrices $\tilde{S}(E) = \hat{\mathbf{I}} + \tilde{\vartheta}(E)(S(E+i0) - \hat{\mathbf{I}})\tilde{\vartheta}(E)$ and $\tilde{S}^\dagger(E) = \hat{\mathbf{I}} + \tilde{\vartheta}(E)(S^\dagger(E+i0) - \hat{\mathbf{I}})\tilde{\vartheta}(E)$. It follows from Eq. (4.13) that operator $\mathcal{T}$ may be considered as a kind of "multichannel T–matrix" (cf. Ref. [39]) for the system of three particles.

Note that the matrix $\mathcal{T}(z)$ may be replaced in Eq. (4.13) with the matrix $\mathcal{T}^\dagger(z)$ obtained from $\mathcal{T}(z)$ by the substitution $\Upsilon\mathbf{v} \to \mathbf{v}\Upsilon$ (respectively, $U \to U^\dagger = \mathbf{v}\Upsilon + \Upsilon M \Upsilon$) in the second component of the lower row of (4.6). To prove that $(\mathbf{J}\mathcal{T}^\dagger\mathbf{J}^\dagger)(z) = (\mathbf{J}\mathcal{T}\mathbf{J}^\dagger)(z)$, it is sufficient to observe that for $z = E \pm i0$, $E > \lambda_{\alpha,j}$, $\alpha = 1,2,3$, $\quad j = 1,2,...,n_\alpha$,

$$(\mathrm{J}_1\Psi^*\Upsilon\mathbf{v}\Psi\mathrm{J}_1^\dagger)(z) = (\mathrm{J}_1\Psi^*\mathbf{v}\Upsilon\Psi\mathrm{J}_1^\dagger)(z). \tag{4.14}$$

Indeed, according to Eqs. (3.4) and (3.5),

$$(\Psi^*\Upsilon\mathbf{v}\Psi)_{\alpha,i;\beta,j}(p_\alpha, p'_\beta) = -\frac{1-\delta_{\alpha\beta}}{|s_{\alpha\beta}|^3} \cdot \frac{\overline{\phi}_{\alpha,i}(\tilde{k}_\alpha^{(\beta)}(p_\alpha, p'_\beta))\;\phi_{\beta,j}(\tilde{k}_\beta^{(\alpha)}(p'_\beta, p_\alpha))}{[\tilde{k}_\alpha^{(\beta)}(p_\alpha, p'_\beta)]^2 - \lambda_{\alpha,i}}, \tag{4.15}$$

$$(\Psi^*\mathbf{v}\Upsilon\Psi)_{\alpha,i;\beta,j}(p_\alpha, p'_\beta) = -\frac{1-\delta_{\beta\alpha}}{|s_{\alpha\beta}|^3} \cdot \frac{\overline{\phi}_{\alpha,i}(\tilde{k}_\alpha^{(\beta)}(p_\alpha, p'_\beta))\;\phi_{\beta,j}(\tilde{k}_\beta^{(\alpha)}(p'_\beta, p_\alpha))}{[\tilde{k}_\beta^{(\alpha)}(p'_\beta, p_\alpha)]^2 - \lambda_{\beta,j}} \tag{4.16}$$

where

$$\tilde{k}_\gamma^{(\delta)}(q,q')) = \frac{-c_{\gamma\delta}q + q'}{s_{\gamma\delta}}, \quad \gamma,\delta = 1,2,3, \tag{4.17}$$

$q, q' \in \mathbf{R}^3$ (we shall suppose later that $q, q \in \mathbf{C}^3$). One can easily to understand that on the energy shells $|p_\alpha| = \sqrt{E - \lambda_{\alpha,i}}$, $\quad |p'_\beta| = \sqrt{E - \lambda_{\beta,j}}$, $\quad E > \lambda_{\alpha,i}$, $\quad E > \lambda_{\beta,j}$, the denominators of the fractions (4.15) and (4.16) coincide,

$$\begin{aligned}(\tilde{k}_\alpha^{(\beta)})^2 - \lambda_{\alpha,i} &= (\tilde{k}_\beta^{(\alpha)})^2 - \lambda_{\beta,j} = \\ &= \frac{1}{|s_{\alpha\beta}|^2}(E - \lambda_{\alpha,i} + E - \lambda_{\beta,j} - 2c_{\alpha\beta}\sqrt{E-\lambda_{\alpha,i}}\sqrt{E-\lambda_{\beta,j}}(\hat{p}_\alpha, \hat{p}'_\beta) - s_{\alpha\beta}^2 E).\end{aligned} \tag{4.18}$$

Meanwhile the expression (4.18) can not become zero at $E > \lambda_{\alpha,i}$, $E > \lambda_{\beta,j}$ (see Lemma 2). It follows now from Eqs. (4.15), (4.16) and (4.18) that the equality (4.14) is true.

Along with $S(z)$ and $S^{\dagger}(z)$ we shall consider also the *truncated* scattering matrices

$$S_l(z) \equiv \hat{\mathbf{I}} + (\tilde{L}\mathbf{J}\mathcal{T}\mathbf{J}^{\dagger}LA)(z) \ \text{ and } \ S_l^{\dagger}(z) \equiv \hat{\mathbf{I}} + (AL\mathbf{J}\mathcal{T}\mathbf{J}^{\dagger}\tilde{L})(z), \tag{4.19}$$

where the multi–index

$$l = (l_0, l_{1,1}, ..., l_{1,n_1}, l_{2,1}, ..., l_{2,n_2}, l_{3,1}, ..., l_{3,n_3}) \tag{4.20}$$

has the components $l_0 = 0$ or $l_0 = \pm 1$ and $l_{\alpha,j} = 0$ or $l_{\alpha,j} = 1$, $\alpha = 1, 2, 3, \quad j = 1, 2, ..., n_\alpha$. By L and $\tilde{L}$ we denote the diagonal number matrices

$$L = \operatorname{diag}\{l_0, l_{1,1}, ..., l_{1,n_1}, l_{2,1}, ..., l_{2,n_2}, l_{3,1}, ..., l_{3,n_3}\} \tag{4.21}$$

and

$$\tilde{L} = \operatorname{diag}\{|l_0|, l_{1,1}, ..., l_{1,n_1}, l_{2,1}, ..., l_{2,n_2}, l_{3,1}, ..., l_{3,n_3}\}, \tag{4.22}$$

corresponding to the multi–index l. The matrix $\tilde{L}$ is evidently to be a projector in $\hat{\mathcal{H}}_0 \oplus \hat{\mathcal{H}}_1$ on the subspace $\hat{\mathcal{H}}_1^{(l)}$ if $l_0 = 0$ or on the subspace $\hat{\mathcal{H}}_0 \oplus \hat{\mathcal{H}}_1^{(l)}$ if $l_0 \neq 0$. Here in both cases, $\hat{\mathcal{H}}_1^{(l)} = \bigoplus\limits_{l_{\alpha,j}\neq 0} \hat{\mathcal{H}}^{(\alpha,j)}$.

As can be seen from formulas (4.13) and (4.6) the scattering matrices $S(z)$ and $S^{\dagger}(z)$ include kernels $M_{\alpha\beta}(P, P', z)$ taken on the energy shells: their arguments $P \in \mathbf{R}^6$ and $P' \in \mathbf{R}^6$ are connected with the energy $z = E + i0$ by Eqs. (2.11) at $E > 0$ or (2.12) at $E > \lambda_{\alpha,j}$. We establish below [see formula (6.8)] that analytical continuation of the matrix $M(z)$ on unphysical sheets of energy z is expressed in terms of analytical continuation of the truncated scattering matrices $S_l(z)$ or $S_l^{\dagger}(z)$ and the half–on–shell Faddeev components $M_{\alpha\beta}(z)$ taken on the physical sheet. More precisely, along with $S_l(z)$, the final formula (6.8) includes the matrices $(L_0\mathbf{J}_0M)(z)$, $(L_1\mathrm{J}_1\Psi^*\Upsilon M)(z)$ and $(M\mathbf{J}_0^{\dagger}L_0)(z)$, $(M\Upsilon\Psi\mathrm{J}_1^{\dagger}L_1)(z)$. Here, l is a certain multi–index (4.20) and $L = \operatorname{diag}\{L_0, L_1\}$ is the respective matrix (4.21) with $L_0 = l_0$.

In the rest of this section we shall formulate some statements (Theorems 3–7) concerning the existence of the analytical continuation of the above matrices and their domains of holomorphness. In view of shortage of space we shall not give here full proofs. Note only that proofs are based on analysis [24] of the Faddeev equations (2.6). For all this, one has additionally to pay a special attention to studying the domains of holomorphness in z of the functions

$$\left[p_\alpha^2 + {p'_\beta}^2 - 2c_{\alpha\beta}(p_\alpha, p'_\beta) - s_{\alpha\beta}^2 z\right]^{-1}, \tag{4.23}$$

with one or both arguments p_α and p'_β situating on the energy shells (2.11) or (2.12). Functions (4.23) arise when iterating Eq. (2.6) because of the presence of the multiplier $\mathbf{R}_0$ in the operator $-\mathbf{t}\mathbf{R}_0\Upsilon$. Also, the functions (4.23) appear as a display of singularities (3.5) of the eigenfunctions $\psi_{\alpha,j}$, $\alpha = 1, 2, 3, \quad j = 1, 2, ..., n_\alpha$.

In the case when the arguments p_α and/or p'_β are taken on the shells (2.12), $p_\alpha = \sqrt{z - \lambda_{\alpha,i}}\,\hat{p}_\alpha$ and $p'_\beta = \sqrt{z - \lambda_{\beta,j}}\,\hat{p}'_\alpha$, the holomorphness domains of the functions (4.23) with respect to the variable z are described by the following plain lemmas.

Lemma 1. *For any $\rho \geq 0$, $-1 \leq \eta \leq 1$, the domain*

$$\operatorname{Re} z > \frac{\lambda}{c^2} + \frac{c^2}{4s^2|\lambda|}(\operatorname{Im} z)^2 \tag{4.24}$$

contains no roots z of the equation

$$z - \lambda + \rho + 2c\sqrt{z - \lambda}\sqrt{\rho}\,\eta - s^2 z = 0, \tag{4.25}$$

with $\lambda<0$, $0<|c|<1$ and $s^2=1-c^2$. For any number $z\in\mathbf{C}$ outside the domain (4.24) one can always find such values of parameters $\rho\ge 0$ and η, $-1\le\eta\le 1$, that the left–hand part of Eq. (4.25) becomes equal to zero at the point z.

Lemma 2. *Let the parameters of the equation*

$$z-\lambda_1+z-\lambda_2+2c\sqrt{z-\lambda_1}\sqrt{z-\lambda_2}\,\eta-s^2z=0 \tag{4.26}$$

be such: $\eta\in[-1,\,1]$, $\lambda_1\le\lambda_2<0$, $0<c<1$ and $s^2=1-c^2$. Then the following assertions take place.

1) If $|\lambda_2|>c^2|\lambda_1|$ then for all $\eta\in[-1,1]$ Eq. (4.26) has a unique root z and this root is real. Moreover $z=z_+$ if $\eta\ge 0$, and $z=z_-$ if $\eta\le 0$ with

$$z_\pm=\frac{(1+c^2-2c^2\eta^2)(\lambda_1+\lambda_2)\pm 2\sqrt{c^2\eta^2[\lambda_1\lambda_2\,s^4-(\lambda_2-\lambda_1)^2c^2(1-\eta^2)]}}{(1+c^2)^2-4c^2\eta^2}. \tag{4.27}$$

When η runs the interval $[-1,1]$, the roots $z_\pm$ fill the interval $[z_{\rm lt},z_{\rm rt}]$ with the ends

$$z_{\rm lt}=\frac{1}{s^2}[-|\lambda_1|-|\lambda_2|-2c\sqrt{|\lambda_1|\cdot|\lambda_2|}] \tag{4.28}$$

and

$$z_{\rm rt}=\frac{1}{s^2}[-|\lambda_1|-|\lambda_2|+2c\sqrt{|\lambda_1|\cdot|\lambda_2|}],\quad z_{\rm rt}<\lambda_1. \tag{4.29}$$

2) If $|\lambda_2|=c^2|\lambda_1|$ then Eq. (4.26) has two real roots:

a) the root $z=\lambda_1$ existing for all $\eta\in[-1,1]$;

b) the root $z=z_-$ given by (4.27) which exists for $-1\le\eta\le 0$ only.

For $-1\le\eta\le 1$ these roots together fill the interval $[z_{\rm lt},\lambda_1]$ with $z_{\rm lt}=-|\lambda_1|\,(1+2c^4/s^2)$.

3) If $|\lambda_2|<c^2|\lambda_1|$ then

a) for $-1\le\eta\le\eta^*$, $\eta^*=\dfrac{\sqrt{c^2-\rho}\sqrt{1-c^2\rho}}{c(1-\rho)}$, $\rho=\dfrac{|\lambda_2|}{|\lambda_1|}$, Eq. (4.26) has two real roots $z_\pm$ given by (4.27), which fill the interval $[z_{\rm lt},z_{\rm rt}]$ with the ends (4.28) and (4.29), $z_{\rm rt}<\lambda_1$;

b) for $\eta^*<\eta\le 0$ Eq. (4.26) has two complex roots $z_\pm$ described again by Eq. (4.27). When η moves, these roots fill the ellipse centered in the point $z_c=-|\lambda_1|\left[1+\dfrac{(c^2-\rho)^2}{s^2(1+c^2)(1+\rho)}\right]$. Half–axes of the ellipse are given by $a=|\lambda_1|\cdot\dfrac{(c^2-\rho)(1-c^2\rho)}{(1+c^2)s^2(1+\rho)}$ (along real axis) and $b=|\lambda_1|\cdot\dfrac{(c^2-\rho)(1-c^2\rho)}{(1+c^2)s^2(1-\rho)\sqrt{(1+c^2)^2-4c^2\eta^{*2}}}$ (along imaginary axis). The right vertex of the ellipse is located in the point $z_{\rm rt}^{(\rm e)}=z_c+a=-\dfrac{|\lambda_1|+|\lambda_2|}{1+c^2}$ situated between λ_1 and λ_2. Its left vertex is $z_{\rm lt}^{(\rm e)}=z_c-a<z_{\rm rt}$.

Let $\Pi_b^{(\beta,j)}$ be the domain in the complex plane $\mathbf{C}$ with cut along the ray $[\lambda_{\min},+\infty)$ where the conditions (4.24) with $\lambda=\lambda_{\beta,j}$, $c=c_{\alpha\beta}$ and the inequalities

$$\operatorname{Re}z>\lambda_{\beta,j}-s_{\alpha\beta}^2b^2+\frac{1}{4s_{\alpha\beta}^2b^2}(\operatorname{Im}z)^2 \tag{4.30}$$

are valid simultaneously for all $\alpha=1,2,3$, $\alpha\neq\beta$. In the case of the potentials (2.2) one has to take $b=+\infty$ in (4.30).

By $\mathcal{R}_{\alpha,i;\beta,j}$, $\alpha\neq\beta$, we denote domain complementary in $\mathbf{C}\setminus[\lambda_{\min},+\infty)$ to the set filled by the roots of Eq. (4.26) in the case when $\lambda_1=\min\{\lambda_{\alpha,i},\lambda_{\beta,j}\}$, $\lambda_2=\max\{\lambda_{\alpha,i},\lambda_{\beta,j}\}$, $c=|c_{\alpha\beta}|$ and $\eta=(\hat p_\alpha,\hat p'_\beta)$ runs the interval $[-1,1]$.

Theorem 3. *The matrix integral operator $L_1'\hat{\mathcal{T}}_{11}(z)L_1''$, $z = E \pm i0$, acting in $\hat{\mathcal{H}}_1$, allows analytical continuation in z from rims of the ray $E \in (\lambda, +\infty)$, $\lambda = \max\limits_{\substack{l'_{\gamma,k}\neq 0,\\ l''_{\gamma,k}\neq 0}} \lambda_{\gamma,k}$, on the domain*

$$\Pi^{(\mathrm{hol})}_{l'l''} = [\bigcap_{\substack{l'_{\alpha,i}\neq 0\\ l''_{\beta,j}\neq 0}} \mathcal{R}_{\alpha,i;\,\beta,j}]\bigcap[\bigcap_{\substack{l'_{\gamma,k}\neq 0,\\ l''_{\gamma,k}\neq 0}} \Pi_b^{(\gamma,k)}] \setminus \overline{\sigma(H)} \tag{4.31}$$

where $l_1' = \mathrm{diag}\left(l_0', l_{1,1}', ..., l_{1,n_1}',\ l_{2,1}', ..., l_{2,n_2}',\ l_{3,1}', ..., l_{3,n_3}'\right)$, $l_1'' = \mathrm{diag}\left(l_0'', l_{1,1}'', ..., l_{1,n_1}'',\ l_{2,1}'', ..., l_{2,n_2}'', l_{3,1}'', ..., l_{3,n_3}''\right)$, with $l_0' = l_0'' = 0$. The nontrivial kernels $\left(L_1'\hat{\mathcal{T}}_{11}(z)L_1''\right)_{\alpha,i;\,\beta,j}(\hat{p}_\alpha, \hat{p}_\beta', z)$, $l_{\alpha,i}' \neq 0$, $l_{\beta,j}'' \neq 0$, turn into functions holomorphic concerning $z \in \Pi^{(\mathrm{hol})}_{l'l''}$ and real–analytic with respect to $\hat{p}_\alpha, \hat{p}_\beta' \in S^2$.

Remark 1. The domains $\Pi^{(\mathrm{hol})}_{l'l''}$ and $\Pi^{(\mathrm{hol})}_{l''l'}$ coincide, $\Pi^{(\mathrm{hol})}_{l'l''} = \Pi^{(\mathrm{hol})}_{l''l'}$.

If $l' = l'' = l$, we use for $\Pi^{(\mathrm{hol})}_{l'l''}$ the notation $\Pi^{(\mathrm{hol})}_{l}$,

$$\Pi^{(\mathrm{hol})}_{l} = \Pi^{(\mathrm{hol})}_{ll}. \tag{4.32}$$

Theorem 4. *Let $L_0 = l_0 = 0$. Then the matrices $(M\Upsilon\Psi \mathrm{J}_1^\dagger L_1)(z)$ and $(L_1 \mathrm{J}_1 \Psi^* \Upsilon M)(z)$, $z = E \pm i0$, allow analytical continuation in z from rims of the ray $E \in (\lambda, +\infty)$, $\lambda = \max\limits_{(\beta,j):\, l_{\beta,j}\neq 0} \lambda_{\beta,j}$, on the domain $\Pi^{(\mathrm{hol})}_{l} \setminus \overline{\sigma(H)}$ as a bounded for $z \notin [\lambda_{\min}, +\infty)$ operator–valued functions of variable z, $(M\Upsilon\Psi \mathrm{J}_1^\dagger L_1)(z)$: $\hat{\mathcal{H}}_1 \to \mathcal{G}_0$ and $(L_1 \mathrm{J}_1 \Psi^* \Upsilon M)(z)$: $\mathcal{G}_0 \to \hat{\mathcal{H}}_1$.*

Continuing the half–on–shell matrices $(\mathbf{J}_0 M)(z)$, $(M\mathbf{J}_0^\dagger)(z)$, $z = E \pm i0$, $E > 0$, into domain of complex z is considered in a sense of distributions over $\mathcal{O}(\mathbf{C}^6)$. For example of $M\mathbf{J}_0^\dagger$ we consider continuation of the bilinear form

$$(F,\,(M\mathbf{J}_0^\dagger)(E \pm i0)) \equiv \sum_{\alpha,\beta} \int_{\mathbf{R}^6} dP \int_{S^5} d\hat{P}'\, F_\alpha(P) M_{\alpha\beta}(P, \pm\sqrt{E}\,\hat{P}', E \pm i0)\, f_\beta(\hat{P}')$$

where $F = (F_1, F_2, F_3)$ with $F_\alpha \in \mathcal{O}(\mathbf{C}^6)$ and $f = (f_1, f_2, f_3)$ with $f_\alpha \in \hat{\mathcal{H}}_0$.

When constructing continuation of this form and that for $(\mathbf{J}_0 M)(E \pm i0)$ we base on two simple statements concerning the domains of holomorphness of the function (4.23) in the case when argument P' belongs to the three–body energy shell (2.11) and therefore $p_\beta' = \sqrt{z}\nu'\hat{p}_\beta'$ with $\nu' \in [0,\,1]$.

Lemma 3. *Let in the equation $\rho + z\nu' + 2c\sqrt{z}\sqrt{\nu'}\sqrt{\rho}\eta - s^2 z = 0$, the parameters ν' and η run the intervals $0 \le \nu' \le 1$ and $-1 \le \eta \le 1$ respectively, and $c > 0$, $s^2 = 1 - c^2$, $z \in \mathbf{C}$ be fixed. Then the roots ρ of the above equation fill the set consisting of the line segment $[0, z]$ on the complex plane $\mathbf{C}$ and the circle centered in the origin, the radius of which being equal to $c^2|z|$.*

Lemma 4. *Let the parameters of the equation*

$$z - \lambda + z\nu + 2c\sqrt{z}\sqrt{z-\lambda}\sqrt{\nu}\eta - s^2 z = 0, \tag{4.33}$$

satisfy the conditions $\nu \in [0, 1]$, $\eta \in [-1, 1]$, $\lambda < 0$, $c \in (0, 1)$ and $s^2 = 1 - c^2$. Then if ν and η run the above ranges, the roots z of Eq. (4.33) fill the ray $(-\infty, \lambda/(1 + c^4)]$ and the circle centered in the point $z_c = \lambda/(1 - c^4)$, radius of which equals to $c^2\lambda/(1 - c^4)$.

Let $\tilde{\Pi}_b^{(0)\pm}$, $\tilde{\Pi}_b^{(0)\pm}\subset \mathbf{C}^\pm$, be the domains complementary in $\mathbf{C}^\pm$ to the totality of circles having radii $r=c_{\alpha\beta}^2|\lambda_{\alpha,j}|/(1-c_{\alpha\beta}^4)$ and centered in the points $z_c=\lambda_{\alpha,j}/(1-c_{\alpha\beta}^4)$ where $\alpha,\beta=1,2,3,\ \beta\neq\alpha$, and $j=1,2,...,n_\alpha$. In the case of the potentials (2.3) the domains $\tilde{\Pi}_b^{(0)\pm}$ must satisfy extra conditions

$$\operatorname{Re} z > -\frac{|s_{\alpha\beta}|^2 b^2}{(1+|c_{\alpha\beta}|)^2}+\frac{(1+|c_{\alpha\beta}|)^2}{4|s_{\alpha\beta}|^2 b^2}\,(\operatorname{Im} z)^2. \tag{4.34}$$

for all $\alpha,\beta=1,2,3,\ \beta\neq\alpha$.

Utilizing Lemmas 3 and 4 one can prove the following

Theorem 5. *Kernels of the matrices $(M\mathbf{J}_0^\dagger)(z)$ and $(\mathbf{J}_0 M)(z)$, $z=E\pm i0$, $E>0$, allow analytical continuation in z on the domains, respectively $\tilde{\Pi}_b^{(0)+}$ and $\tilde{\Pi}_b^{(0)-}$, $\tilde{\Pi}_b^{(0)\pm}\subset\mathbf{C}^\pm$. The continuation of kernels of the matrices $\left(\mathcal{Q}^{(n)}\mathbf{J}_0^\dagger\right)(z)$, and $\left(\mathbf{J}_0\mathcal{Q}^{(n)}\right)(z)$, $n\le 3$, included in the representation (4.5) for $M(z)$ has to be understood in a sense of distributions over $\mathcal{O}(\mathbf{C}^6)$. At the same time the kernels*

$$\begin{gathered}\mathcal{F}_{\alpha\beta}(P,\sqrt{z}\hat{P}',z),\ \mathcal{I}_{\alpha,j;\beta}(p_\alpha,\sqrt{z}\hat{P}',z),\\ \mathcal{J}_{\alpha;\beta,k}(P,\sqrt{z}\sqrt{\nu'}\,\hat{p}'_\beta,z)\ \ \text{and}\ \ \mathcal{K}_{\alpha,j;\beta,k}(p_\alpha,\sqrt{z}\sqrt{\nu'}\,\hat{p}'_\beta,z)\end{gathered} \tag{4.35}$$

$$\alpha,\beta=1,2,3,\ j=1,2,...,n_\alpha,\ k=1,2,...,n_\beta,$$

of the matrices $(\mathcal{Q}^{(n)}\mathbf{J}_0^\dagger)(z)$, $n\ge 4$, and $(\mathcal{W}\mathbf{J}_0^\dagger)(z)$ as well as the kernels

$$\begin{gathered}\mathcal{F}_{\alpha\beta}(\sqrt{z}\hat{P},P',z),\ \mathcal{I}_{\alpha,j;\beta}(\sqrt{z}\sqrt{\nu}\,\hat{p}_\alpha,P',z),\\ \mathcal{J}_{\alpha;\beta,k}(\sqrt{z}\hat{P},p'_\beta,z)\ \ \text{and}\ \ \mathcal{K}_{\alpha,j;\beta,k}(\sqrt{z}\sqrt{\nu}\,\hat{p}_\alpha,p'_\beta,z)\end{gathered} \tag{4.36}$$

of the matrices $(\mathbf{J}_0\mathcal{Q}^{(n)})(z)$, $n\ge 4$, and $(\mathbf{J}_0\mathcal{W})(z)$ can be continued on the domains $\tilde{\Pi}_b^{(0)\pm}$ as usual holomorphic functions of variable z. Being Hölder functions of variables $\hat{P}'\in S^5$ or $\sqrt{\nu'}\hat{p}'_\beta$, $0\le\nu'\le 1$, $\hat{p}'_\beta\in S^2$ ($\hat{P}\in S^5$ or $\sqrt{\nu}\hat{p}_\alpha$, $0\le\nu\le 1$, $\hat{p}_\alpha\in S^2$) with index $\mu'\in(0,\ 1/8)$, the kernels (4.35) (kernels (4.36)) considered as functions of $P\in\mathbf{R}^6$, $p_\alpha\in\mathbf{R}^3$ ($P'\in\mathbf{R}^6$, $p'_\beta\in\mathbf{R}^3$), can be embedded in their totality in $\mathcal{B}(\theta,\mu)$ with θ and μ, the arbitrary numbers such that $\theta\in(3/2,\ \theta_0)$ and $\mu\in(0,\ 1/8)$. At $|\operatorname{Im} z|\ge\delta>0$ one can take $\mu=1$.

Let us comment the assertion of the theorem for example of the matrices $(M\mathbf{J}_0^\dagger)(z)$. Note in particular that continuation on $\tilde{\Pi}_b^{(0)\pm}$ of the form $\left(F,(\mathcal{Q}^{(0)}\mathbf{J}_0^\dagger)(z)f\right)=\sum_\alpha\left(F_\alpha,(\mathbf{t}_\alpha\mathbf{J}_0^\dagger)(z)f_\alpha\right)$ is described by the equalities

$$\begin{aligned}(F_\alpha,(\mathbf{t}_\alpha \mathrm{J}_0^\dagger)(z)f_\alpha)\equiv\int\limits_{\mathbf{R}^3}dk_\alpha\int\limits_{S^2}d\hat{k}'_\alpha\int\limits_{S^2}d\hat{p}'_\alpha\int\limits_0^{\pi/2}d\omega'_\alpha\sin^2\omega'_\alpha\cos^2\omega'_\alpha\times\\ \times t_\alpha(k_\alpha,\sqrt{z}\cos\omega'_\alpha\hat{k}'_\alpha,z\cos^2\omega'_\alpha)F_\alpha(k_\alpha,\pm\sqrt{z}\sin\omega'_\alpha\hat{p}'_\alpha)\cdot f_\alpha(\omega'_\alpha,\hat{k}'_\alpha,\hat{p}'_\alpha),\end{aligned} \tag{4.37}$$

where $\omega'_\alpha,\hat{k}'_\alpha,\hat{p}'_\alpha$ are the hyperspherical coordinates [29] of the point $\hat{P}'\in S^5$, $\omega'_\alpha\in[0,\pi/2]$, $\hat{k}'_\alpha,\hat{p}'_\alpha\in S^2$. Note also that $\hat{P}'=\{\cos\omega'_\alpha\hat{k}'_\alpha,\ \sin\omega'_\alpha\hat{p}'_\alpha\}$ and $d\hat{P}'=\sin^2\omega'_\alpha\cos^2\omega'_\alpha d\omega'_\alpha d\hat{k}'_\alpha d\hat{p}'_\alpha$ is a measure on S^5.

The analytical continuation on $\tilde{\Pi}_b^{(0)\pm}$ of the form $\left(F,(\mathcal{Q}^{(1)}\mathbf{J}_0^\dagger)(E\pm i0)f\right)$ is given by

$$\left(F,(\mathcal{Q}^{(1)}\mathbf{J}_0^\dagger)(z)f\right)=\sum_{\alpha,\beta,\ \alpha\neq\beta}Q^\pm_{1,\alpha\beta}(z)+Q^\pm_{2,\alpha\beta}(z) \tag{4.38}$$

where

$$Q^{\pm}_{1,\alpha\beta}(z) = \pm\frac{\sqrt{z}}{4}\frac{1}{|s_{\alpha\beta}|}\int\limits_{\mathbf{R}^3} dk_\alpha \int\limits_{S^2} d\hat{p}_\alpha \int\limits_{S^2} dk'_\beta \int\limits_{S^2} d\hat{p}'_\beta \int\limits_0^1 d\nu\,\sqrt{\nu}\cdot\int\limits_0^1 d\nu'\,\sqrt{\nu'}\sqrt{1-\nu'}\times$$

$$\times\frac{F_\alpha(k_\alpha,\sqrt{z}\sqrt{\nu}\hat{p}_\alpha)\cdot f_\beta(\sqrt{1-\nu'}\hat{k}'_\beta,\sqrt{\nu'}\hat{p}'_\beta)}{\nu+\nu'-2c_{\alpha\beta}\sqrt{\nu}\sqrt{\nu'}\,(\hat{p}_\alpha,\hat{p}'_\beta)-s^2_{\alpha\beta}\mp i0}\times \tag{4.39}$$
$$\times\, t_\alpha(k_\alpha,\tilde{k}^{(\beta)}_\alpha(\sqrt{z}\sqrt{\nu}\hat{p}_\alpha,\sqrt{z}\sqrt{\nu'}\hat{p}'_\beta),z(1-\nu))\,\times$$
$$\times\, t_\beta(\tilde{k}^{(\alpha)}_\beta(\sqrt{z}\sqrt{\nu'}\hat{p}'_\beta,\sqrt{z}\sqrt{\nu}\hat{p}_\alpha),\sqrt{z}\sqrt{1-\nu'}\hat{k}'_\beta,z(1-\nu'))$$

and

$$Q^{\pm}_{2,an\beta}(z) = \pm\frac{1}{4}\cdot\frac{1}{|s_{\alpha\beta}|}\int\limits_{\mathbf{R}^3} dk_\alpha \int\limits_{S^2} d\hat{p}_\alpha \int\limits_{S^2} dk'_\beta \int\limits_{S^2} d\hat{p}'_\beta \int\limits_{\Gamma^{\pm}_z} d\rho\,\sqrt{\rho}\cdot\int\limits_0^1 d\nu'\,\sqrt{\nu'}\sqrt{1-\nu'}\times$$

$$\times\frac{F_\alpha(k_\alpha,\pm\sqrt{\rho}\hat{p}_\alpha)\cdot f_\beta(\sqrt{1-\nu'}\hat{k}'_\beta,\sqrt{\nu'}\hat{p}'_\beta)}{\rho+z\nu'-2c_{\alpha\beta}\sqrt{z}\sqrt{\rho}\sqrt{\nu'}\,(\hat{p}_\alpha,\hat{p}'_\beta)-s^2_{\alpha\beta}z}\times \tag{4.40}$$
$$\times\, t_\alpha(k_\alpha,\tilde{k}^{(\beta)}_\alpha(\pm\sqrt{\rho}\hat{p}_\alpha,\sqrt{z}\sqrt{\nu'}\hat{p}'_\beta),z-\rho)\,\times$$
$$\times\, t_\beta(\tilde{k}^{(\alpha)}_\beta(\sqrt{z}\sqrt{\nu'}\hat{p}'_\beta,\pm\sqrt{\rho}\hat{p}_\alpha),\sqrt{z}\sqrt{1-\nu'}\hat{k}'_\beta,z(1-\nu')).$$

Here, by Γ^+_z (Γ^-_z) we understand a path of integration beginning at z and going clockwise (counterclockwise) along the circumference $C_{|z|}$ having radius $|z|$ and centered in the origin. After the path crosses the real axis, it goes further along this one so that the rest of Γ^+_z (Γ^-_z) consists of the points $\rho=\lambda+i0$ $(\rho=\lambda+i0)$, $\lambda\in(|z|,+\infty)$.

Boundaries of the holomorphness domains $\tilde{\Pi}^{(0)\pm}_b$ of the form $\left(F,(\mathcal{Q}^{(1)}\mathbf{J}^\dagger_0)(z)f\right)$ are found as a matter of the fact, from those requirements that the poles of T–matrices $t_\alpha(\,\cdot\,,\,\cdot\,,z(1-\nu))$ and $t_\beta(\,\cdot\,,\,\cdot\,,z(1-\nu'))$ which are present in the integral (4.39), have not to manifest itself in above domains. Also, we require the same from the poles of T–matrices $t_\alpha(\,\cdot\,,\,\cdot\,,z-\rho)$ which are present in the integral (4.40). If $z\notin(-\infty,\lambda_{\max}]$ then the appearance conditions $z(1-\nu)=\lambda_{\alpha,j}$, $j=1,2,...,n_\alpha$, $z(1-\nu')=\lambda_{\beta,k}$, $k=1,2,...,n_\beta$, for the poles of the T–matrices $t_\alpha(\,\cdot\,,\,\cdot\,,z(1-\nu))$ and $t_\beta(\,\cdot\,,\,\cdot\,,z(1-\nu'))$, are valid for no $\nu,\nu'\in[0,1]$. The appearance conditions $z-\rho=\lambda_{\alpha,j}$, $j=1,2,...,n_\alpha$, of the poles of $t_\alpha(\,\cdot\,,\,\cdot\,,z-\rho)$ may be realized if only the contours $\Gamma^\pm_z$ include into itself more than one fourth of the circumference $C_{|z|}$. However their contribution to $Q^\pm_{2,\alpha\beta}(z)$ arising when the points $\rho=z-\lambda_{\alpha,j}$ cross contours $\Gamma^\pm_z$, may be always taken into account using the residue theorem. We shall not present here respective formulae. Note only that taking of residues in the points $\rho=z-\lambda_{\alpha,j}$ transforms the minor three–body pole singularities of the integrand of $Q^\pm_{2,\alpha\beta}(z)$ into those of the type $(z-\lambda_{\alpha,j}+z\nu'-2c_{\alpha\beta}\sqrt{z}\sqrt{z-\lambda_{\alpha,j}}\sqrt{\nu'}\eta-s^2_{\alpha\beta}z)^{-1}$. Location of such singularities is described by Lemma 4.

The iteration $\mathcal{Q}^{(2)}(z)$ kernels $\mathcal{F}_{\alpha\beta}(P,P',z)$, $\mathcal{I}_{\alpha,j;\beta}(p_\alpha,P',z)$, $\mathcal{J}_{\alpha;\beta,k}(P,p'_\beta,z)$, and $\mathcal{K}_{\alpha,j;\beta,k}(p_\alpha,p'_\beta,z)$, $P,P'\in\mathbf{R}^6$, $p_\alpha,p'_\beta\in\mathbf{R}^3$, have more weak singularities [24], [29] than the $\mathcal{Q}^{(1)}(z)$ components. When continuing the form $\left(F,(\mathcal{Q}^{(2)}\mathbf{J}^\dagger_0)(z)f\right)$ we get for it the representations which differ from (4.38)–(4.40) mainly in replacement of the distributions $\{z(\nu+\nu'-2c_{\alpha\beta}\sqrt{\nu}\sqrt{\nu'}(\hat{p}_\alpha,\hat{p}'_\beta)-s^2_{\alpha\beta}\mp i0)\}^{-1}$, $0\le\nu\le1$, $0\le\nu'\le1$, with functions singular as

$$\frac{1}{z\,|c_{\alpha\gamma}\nu\hat{p}_\alpha-c_{\beta\gamma}\nu'\hat{p}'_\beta|}\cdot\ln\frac{\sqrt{s^2_{\alpha\gamma}(1-\nu^2)}+\sqrt{s^2_{\beta\gamma}(1-\nu'^2)}+|c_{\alpha\gamma}\nu\hat{p}_\alpha-c_{\beta\gamma}\nu'\hat{p}'_\beta|}{\sqrt{s^2_{\alpha\gamma}(1-\nu^2)}+\sqrt{s^2_{\beta\gamma}(1-\nu'^2)}-|c_{\alpha\gamma}\nu\hat{p}_\alpha-c_{\beta\gamma}\nu'\hat{p}'_\beta|} \tag{4.41}$$

The kernels $\mathcal{F}_{\alpha\beta}(P,P',z)$ $\mathcal{I}_{\alpha,j;\beta}(p_\alpha,P',z)$, $\mathcal{J}_{\alpha;\beta,k}(P,p'_\beta,z)$, and $\mathcal{K}_{\alpha,j;\beta,k}(p_\alpha,p'_\beta,z)$ of the iteration $\mathcal{Q}^{(3)}(z)=\left(-\mathbf{t}(z)\mathbf{R}_0(z)\Upsilon\right)^3\mathbf{t}(z)$ are still singular. Though their singularities are weak, continuation of the kernels $\left(\mathcal{Q}^{(3)}\mathbf{J}_0^\dagger\right)(z)$ on the domains $\tilde{\Pi}_b^{(0)\pm}$ we understand as before in a sense of distributions over $\mathcal{O}(\mathbf{C}^6)$. So, we realize it following the same scheme as for the continuation of $\left(\mathcal{Q}^{(1)}\mathbf{J}_0^\dagger\right)(z)$ and $\left(\mathcal{Q}^{(2)}\mathbf{J}_0^\dagger\right)(z)$.

Theorem 6. *The matrix* $(\mathbf{J}_0 M\mathbf{J}_0^\dagger)(z)$ *(the operator* $(\mathrm{J}_0 T\mathrm{J}_0^\dagger)(z)$*) admits the analytical continuation in* z *from the rims of the cut* $z=E\pm i0$, $E>0$, *on the domains* $\tilde{\Pi}_b^{(0)\pm}\in\mathbf{C}^\pm$ *as a bounded operator in* $\hat{\mathcal{G}}_0$ *(in* $\hat{\mathcal{H}}_0$*). For all this* $(\mathbf{J}_0 M\mathbf{J}_0^\dagger)(z)$, $z\in\tilde{\Pi}_b^{(0)\pm}$, *admits the representation* [cf. (4.5)] $(\mathbf{J}_0 M\mathbf{J}_0^\dagger)(z)=\sum_{n=0}^{3}(\mathbf{J}_0\mathcal{Q}^{(n)}\mathbf{J}_0^\dagger)(z)+(\mathbf{J}_0\mathcal{W}\mathbf{J}_0^\dagger)(z)$. *The operators* $(\mathbf{J}_0\mathcal{Q}^{(0)}\mathbf{J}_0^\dagger)(z)$ *and* $(\mathbf{J}_0\mathcal{Q}^{(1)}\mathbf{J}_0^\dagger)(z)$ *are bounded matrix operators in* $\hat{\mathcal{G}}_0$ *with singular kernels. Having weakly singular kernels the matrices* $(\mathbf{J}_0\mathcal{Q}^{(n)}\mathbf{J}_0^\dagger)(z)$, $n=2,3$, *are compact operators in* $\hat{\mathcal{G}}_0$. *To that end kernels of matrix* $(\mathbf{J}_0\mathcal{W}\mathbf{J}_0^\dagger)(z)$ *are Hölder functions of their arguments with the index* $\mu\in(0,\ 1/8)$.

As a comment to this theorem we present explicit formulae for the kernels of the operators $(\mathbf{J}_0\mathcal{Q}^{(0)}\mathbf{J}_0^\dagger)(z)$ and $(\mathbf{J}_0\mathcal{Q}^{(1)}\mathbf{J}_0^\dagger)(z)$.

The first of them have the form $(\mathbf{J}_0\mathcal{Q}^{(0)}\mathbf{J}_0^\dagger)_{\alpha\beta}(\hat{P},\hat{P}',z)=\delta_{\alpha\beta}(\mathrm{J}_0\mathbf{t}_\alpha\mathrm{J}_0^\dagger)(\hat{P},\hat{P}',z)$, $\alpha,\beta=1,2,3$, where

$$\begin{aligned}(\mathrm{J}_0\mathbf{t}_\alpha\mathrm{J}_0^\dagger)(\hat{P},\hat{P}',z)=t_\alpha(\sqrt{z}\cos\omega_\alpha\hat{k}_\alpha,\sqrt{z}\cos{\omega'}_\alpha\hat{k'}_\alpha,z\cos^2\omega_\alpha)\times\\ \times\,\delta(\sqrt{z}\sin\omega_\alpha\hat{p}_\alpha-\sqrt{z}\sin{\omega'}_\alpha\hat{p}'_\alpha).\end{aligned}\tag{4.42}$$

Here, $\omega_\alpha,\hat{k}_\alpha,\hat{p}_\alpha$ and $\omega'_\alpha,\hat{k}'_\alpha,\hat{p}'_\alpha$ are coordinates of the points $\hat{P}=\{k_\alpha,p_\alpha\}$ and $\hat{P}'=\{k'_\alpha,p'_\alpha\}$ on hypersphere S^5. We mean here that

$$\delta(\sqrt{z}\sin\omega\hat{p}-\sqrt{z}\sin\omega'\hat{p}')=\mathrm{Sign\ Im}\,z\cdot\frac{\delta(\hat{p},\hat{p}')\delta(\omega-\omega')}{(\sqrt{z})^3\sin^2\omega\cos\omega},\tag{4.43}$$

where $\delta(\hat{p},\hat{p}')$ is the kernel of the identity operator in $L_2(S^2)$. The denominator $(\sqrt{z})^3\sin^2\omega\cos\omega$ of the right–hand part of Eq. (4.43) represents analytical continuation of the Jacobian for respective replacement of variables.

Therefore the operator $(\mathrm{J}_0\mathbf{t}_\alpha\mathrm{J}_0^\dagger)(z)$ acts at $\mathrm{Im}\,z\neq 0$ on $f\in\hat{\mathcal{H}}_0$ as

$$\begin{aligned}\left((\mathrm{J}_0\mathbf{t}_\alpha\mathrm{J}_0^\dagger)(z)f\right)(\hat{P})=\frac{\mathrm{Sign\ Im}\,z}{\left(\sqrt{z}\right)^3}\cdot\int_{S^2}d\hat{k}'_\alpha\,\times\\ \times\,t_\alpha(\sqrt{z}\cos\omega_\alpha\hat{k}_\alpha,\ \sqrt{z}\cos\omega_\alpha\hat{k}'_\alpha,\ z\cos^2\omega_\alpha)f(\cos\omega_\alpha\hat{k}'_\alpha,\sin\omega_\alpha\hat{p}_\alpha).\end{aligned}\tag{4.44}$$

The operators $(\mathbf{J}_0\mathcal{Q}^{(1)}\mathbf{J}_0^\dagger)(z)$, $z\in\tilde{\Pi}_b^{(0)\pm}$, have the kernels

$$(\mathbf{J}_0\mathcal{Q}^{(1)}\mathbf{J}_0^\dagger)_{\alpha\beta}(\hat{P},\hat{P}',z)=\frac{1}{z}\cdot\frac{1-\delta_{\alpha\beta}}{|s_{\alpha\beta}|}\cdot\frac{t_\alpha(k_\alpha,k_\alpha^{(\beta)},z(1-\nu))\ \ t_\beta(k_\beta^{(\alpha)},k'_\beta,z(1-\nu'))}{\nu+\nu'-2c_{\alpha\beta}\sqrt{\nu}\sqrt{\nu'}(\hat{p}_\alpha,\hat{p}'_\beta)-s_{\alpha\beta}^2\mp i0},$$

where $k_\alpha=\sqrt{z}\sqrt{1-\nu}\hat{k}_\alpha$, $k'_\beta=\sqrt{z}\sqrt{1-\nu'}\hat{k}'_\beta$, $k_\alpha^{(\beta)}=\tilde{k}_\alpha^{(\beta)}(\sqrt{z}\sqrt{\nu}\hat{p}_\alpha,\ \sqrt{z}\sqrt{\nu'}\hat{p}'_\beta)$ and $k_\beta^{(\alpha)}=\tilde{k}_\beta^{(\alpha)}(\sqrt{z}\sqrt{\nu'}\hat{p}'_\beta,\ \sqrt{z}\sqrt{\nu}\hat{p}_\alpha)$. At the same time $\nu=\sin^2\omega_\alpha$ and $\nu'=\sin^2\omega'_\beta$.

Main singularities of the kernels $(\mathbf{J}_0\mathcal{Q}^{(2)}\mathbf{J}_0^\dagger)_{\alpha\beta}(\hat{P},\hat{P}',z)$ in $\hat{P}$, $\hat{P}'$ are described by Eqs. (4.41). Singularities of the kernels $(\mathbf{J}_0\mathcal{Q}^{(3)}\mathbf{J}_0^\dagger)_{\alpha\beta}(\hat{P},\hat{P}',z)$ are more weak.

Theorem 7. *The operators* $(\mathbf{J}_0 M\Upsilon\Psi \mathrm{J}_1^{\dagger})(z) : \hat{\mathcal{H}}_1 \to \hat{\mathcal{G}}_0$, $(\mathrm{J}_1\Psi^*\Upsilon M\mathbf{J}_0^{\dagger})(z) : \hat{\mathcal{G}}_0 \to \hat{\mathcal{H}}_1$, $\hat{\mathcal{T}}_{01}(z) : \hat{\mathcal{H}}_1 \to \hat{\mathcal{H}}_0$, *and* $\hat{\mathcal{T}}_{10}(z) : \hat{\mathcal{H}}_0 \to \hat{\mathcal{H}}_1$ *admit the analytical continuation from rims of the cut* $z = E \pm i0$, $E > 0$, *onto the domains* $\Pi_b^{(0)\pm} \subset \mathbf{C}^{\pm}$ *including the points* $z \in \tilde{\Pi}_b^{(0)\pm} \bigcap\limits_{\beta,j} \Pi_b^{(\beta,j)}$ *satisfying the additional conditions*

$$\operatorname{Re} z > \frac{|s_{\beta\gamma}|^2}{(1+|c_{\beta\gamma}|)^2}\,\lambda_{\beta,j} + \frac{(1+|c_{\beta\gamma}|)^2}{4|s_{\beta\gamma}|^2|\lambda_{\beta,j}|}(\operatorname{Im} z)^2.$$

for any $\beta,\gamma = 1,2,3$, $\beta \neq \gamma$, *and* $j = 1,2,...,n_\beta$. *For all* $z \in \Pi_b^{(0)\pm}$ *including the boundary points* $z = E \pm i0$, $E > 0$, *these operators are compact.*

Later, we shall use the notation

$$\Pi_{l^\pm}^{(\mathrm{hol})} \equiv \Pi_b^{(0)\pm} \bigcap \Pi_{l^{(1)}}^{(\mathrm{hol})}, \qquad (4.45)$$

where $l^\pm = (l_0^\pm, l_{1,1}, ..., l_{1,n_1}, l_{2,1}, ..., l_{2,n_2}, l_{3,1}, ..., l_{3,n_3})$ with $l_0^\pm = \pm 1$, $l_{\alpha,j} = 1$, $\alpha = 1,2,3$, $j = 1,2,...,n_\alpha$, and $l^{(1)} = (0,\ l_{1,1}, ..., l_{1,n_1},\ l_{2,1}, ...,\ l_{2,n_2},\ l_{3,1}, ...,\ l_{3,n_3})$ with the same $l_{\alpha,j}$ as $l^\pm$. Remember that the sets $\Pi_{l^{(1)}}^{(\mathrm{hol})} \equiv \Pi_{l^{(1)}l^{(1)}}^{(\mathrm{hol})}$ were defined by Eqs. (4.31).

As follows from Theorems 3, 6 and 7, the total three–body scattering matrix $S(z)$, $z = E \pm i0$, $E > 0$, admits the analytical continuation as a holomorphic operator–valued function, $S(z) : \hat{\mathcal{H}}_0 \oplus \hat{\mathcal{H}}_1 \to \hat{\mathcal{H}}_0 \oplus \hat{\mathcal{H}}_1$, on the domain $\Pi_{l^+}^{(\mathrm{hol})} \subset \mathbf{C}^+$. For any $z \in \Pi_{l^+}^{(\mathrm{hol})}$ the operator $S(z)$ is bounded. In equal degree the same is true for $S^\dagger(z)$.

5. DESCRIPTION OF (PART OF) THE THREE–BODY RIEMANN SURFACE

By the *three–body energy Riemann surface* we mean the Riemann surface of the kernel $R(P,P',z)$ of the Hamiltonian H resolvent $R(z)$ considered as a function of parameter z, the energy of three–body system.

One has to expect this surface as well as that of the free Green function $R_0(P,P',z)$ to consist of infinite number of sheets already because the threshold $z = 0$ is a logarithmic branching point. Actually the Riemann surface of $R(P,P',z)$ is much more complicated than that of $R_0(P,P',z)$ because besides $z = 0$ it has a lot of additional branching points. For example the pair thresholds $z = \lambda_{\alpha,j}$, $\alpha = 1,2,3$, $j = 1,2,...,n_\alpha$, become square root branching points of this surface. Also, the resonances of the pair subsystems turn into such points. Extra branching points are generated by the boundaries of supports of the function (4.23) singularities which were described in Lemmas 1, 2 and 4.

In the present paper we restrict ourselves to consideration of a "small" part of the total three–body Riemann surface for which we succeeded to find the explicit representations expressing analytical continuation of the Green function $R(P,P',z)$, the kernels of the matrix $M(z)$, as well as the scattering matrix $S(z)$, in terms of the physical sheet [see the formulae respectively, (6.8), (6.9) and (6.11)]. Namely, in the Riemann surface of $R(P,P',z)$ we consider two neighboring *"three–body"* unphysical sheets immediately joint with the physical one along the *three–body* branch of continuous spectrum $[0, +\infty)$. Besides, we examine all the *"two–body"* unphysical sheets, i.e. the sheets where parameter z may be carried if the rounds of two–body thresholds $z = \lambda_{\alpha,j}$, $\alpha = 1,2,3$, $j = 1,2,...,n_\alpha$, are permitted but the crossing of the ray $[0, +\infty)$ is forbidden. Evidently, the part of the three–body surface described includes all the sheets neighboring with physical one. The above sheets are of most interest in applications.

A concrete description of the part under consideration we give using the auxiliary vector–function $\mathsf{f}(z) = (\mathsf{f}_0(z),\, \mathsf{f}_1(z),\, \mathsf{f}_2(z),\, \mathsf{f}_3(z))$, where $\mathsf{f}_0(z) = \ln z$ and $\mathsf{f}_\alpha(z)$, $\alpha = 1,2,3$, are again vector–functions, $\mathsf{f}_\alpha(z) = ((z-\lambda_{\alpha,1})^{1/2}, (z-\lambda_{\alpha,2})^{1/2}, ..., (z-\lambda_{\alpha,n_\alpha})^{1/2})$.

The Riemann surface of $\mathsf{f}(z)$ consists of infinite number of the copies of the complex plane $\mathbf{C}'$ cut along the ray $[\lambda_{\min}, +\infty)$. These sheets are sticked together in a suitable way along rims of the cut segments between neighboring points in the set of thresholds $\lambda_{\alpha,j}$, $\alpha = 1,2,3$, $j = 1,2,...,n_\alpha$, and $\lambda_0 = 0$. The sheets $\Pi_{l_0 l_1 l_2 l_3}$ of this surface are identified by the indices of branches of the functions $\mathsf{f}_0(z) = \ln z$ and $\mathsf{f}_{\alpha,j}(z) = (z-\lambda_{\alpha,j})^{1/2}$ in such a manner that l_0 is integer and l_α, $\alpha = 1,2,3$ are multi–indices, $l_\alpha = (l_{\alpha,1}, l_{\alpha,2}, ..., l_{\alpha,n_\alpha})$, $l_{\alpha,j} = 0,1$. For the main branch of the function $\mathsf{f}_{\alpha,j}(z)$, $\alpha = 1,2,3$, $j = 1,2,...,n_\alpha$, we take $l_{\alpha,j} = 0$, and otherwise $l_{\alpha,j} = 1$. In the case if there exist coinciding thresholds i.e. $\lambda_{\alpha,i} = \lambda_{\beta,j}$ at $\alpha \neq \beta$ and/or $i \neq j$ (this means that discrete spectra of the pair Hamiltonian coincide partly though for two pair subsystems or though one of the pair subsystems has a multiple discrete spectrum) then on the each sheet $\Pi_{l_0 l_1 l_2 l_3}$ indices $l_{\alpha,i}$ and $l_{\beta,j}$ coincide, too, $l_{\alpha,i}{=}l_{\beta,j}$. As l_0 we choose the number of the function $\ln z$ branch, $\ln z = \ln|z| + i\varphi_0 + i2\pi l_0$ with φ_0, the argument of z, $z = |z|\mathrm{e}^{i\varphi_0}$, $\varphi_0 \in [0, 2\pi)$. Sheets $\Pi_{l_0 l_1 l_2 l_3}$ are sticked together (along rims of the cut) in such a way that if parameter z going from the sheet $\Pi_{l_0 l_1 l_2 l_3}$ crosses segment of line between two neighboring thresholds $\lambda_{\alpha,i}$ and $\lambda_{\beta,j}$, $\lambda_{\alpha,i} < \lambda_{\beta,j}$ (or $\lambda_{\max}$ and λ_0) than it comes to the sheet $\Pi_{l'_0 l'_1 l'_2 l'_3}$, with indices $l_{\gamma,k}$ corresponding to $\lambda_{\gamma,k} \leq \lambda_{\alpha,i}$ ($\lambda_{\gamma,k} \leq \lambda_{\max}$) which change by 1. For all this if $l_{\gamma,k} = 0$ then $l'_{\gamma,k} = 1$; if $l_{\gamma,k} = 1$ then $l'_{\gamma,k} = 0$. Indices $l_{\gamma,k}$ for $\lambda_{\gamma,k} > \lambda_{\alpha,i}$ and l_0 stay unchanged: $l'_{\gamma,k} = l_{\gamma,k}$, $l'_0 = l_0$. In the case if parameter z crosses the cut on the right from the three–body threshold λ_0 (at $E > \lambda_0$) then all the indices $l_{\gamma,k}$ change as was described above. Besides, the index l_0 changes by 1, too. If at that, z crosses the cut from below up then $l'_0 = l_0 + 1$. Otherwise $l'_0 = l_0 - 1$. Further, by l we denote the multi–index $l = (l_0, l_1, l_2, l_3)$.

Thus, we have described the Riemann surface of the auxiliary vector–function $\mathsf{f}(z)$.

As mentioned above we shall consider only a part of the three–body Riemann surface which will be denoted by $\Re$. We include in $\Re$ all the sheets Π_l of the Riemann surface of the function $\mathsf{f}(z)$ with $l_0 = 0$. Also, we include in $\Re$ the upper half–plane, $\operatorname{Im} z > 0$, of the sheet Π_l with $l_0 = +1$ and the lower half–plane, $\operatorname{Im} z < 0$, of the sheet Π_l with $l_0 = -1$. For these parts we keep the previous notations Π_l, $l_0 = \pm 1$, assuming additionally that cuts are made on them along the rays belonging to the set $Z_{\mathrm{res}} = \bigcup\limits_{\alpha=1}^{3} Z^{(\alpha)}_{\mathrm{res}}$. Here, $Z^{(\alpha)}_{\mathrm{res}} = \{z :\ z = z_r\rho,\ 1 \leq \rho < +\infty,\ z_r \in \sigma^{(\alpha)}_{\mathrm{res}}\}$ is a totality of the rays beginning at the resonance points $z_r \in \sigma^{(\alpha)}_{\mathrm{res}}$ of the subsystem α and going to infinity along the directions $\hat{z}_r = z_r/|z_r|$.

The sheet Π_l for which all the components of the multi–index l are zero, $l_0 = l_{\alpha,j} = 0$, $\alpha = 1,2,3$, $j = 1,2,...,n_\alpha$, is called physical sheet. The unphysical sheets Π_l with $l_0 = 0$ are called two–body sheets since these ones may be reached rounding the two–body thresholds only and it is not necessary to round the three–body threshold λ_0. The sheets Π_l at $l_0 = \pm 1$ are called three–body sheets.

On the base of Sec. 4 results one can prove the following

Lemma 5. *For each two–body unphysical sheet Π_l of the surface $\Re$ there exists such a path from the physical sheet Π_0 to the domain $\Pi_l^{(\mathrm{hol})}$ of Π_l going only on the two–body unphysical sheets $\Pi_{l'}$ that moving by this path, the parameter z stays always in respective domains $\Pi_{l'}^{(\mathrm{hol})} \subset \Pi_{l'}$.*

6. CONTINUATION OF THE FADDEEV EQUATIONS AND REPRESENTATIONS FOR MATRIX $M(z)$, SCATTERING MATRICES AND RESOLVENT ON UNPHYSICAL SHEETS

In the present section we formulate main results of the paper. In view of space shortage their proofs will be given in the following paper [43]. Here we outline only schemes of the proofs.

We begin with description of continuation on unphysical sheets of the Faddeev equations (2.4).

Let $L^{(\alpha)} = \operatorname{diag}\{l_{\alpha,1},\ l_{\alpha,2}, ...,\ l_{\alpha,n_\alpha}\}$ be the diagonal number matrix constructed of the components $l_{\alpha,1},\ l_{\alpha,2}, ...,\ l_{\alpha,n_\alpha}$ of the multi–index l identifying a certain sheet $\Pi_l \subset \Re$. For all this $L_1(l) = \operatorname{diag}\{L^{(1)},\ L^{(2)},\ L^{(3)}\}$ and $L(l) = \operatorname{diag}\{L_0,\ L_1\}$ $\ L_0 \equiv l_0$.

Let $\mathbf{s}_{\alpha,l}(z)$ be the operator defined in $\hat{\mathcal{H}}_0$ by

$$\mathbf{s}_{\alpha,l}(z) = \hat{I}_0 + \mathrm{J}_0(z)\mathbf{t}_\alpha(z)\mathrm{J}_0^\dagger(z)A_0(z)L_0, \quad z \in \Pi_0. \tag{6.1}$$

It follows from Eq. (6.1) that $\mathbf{s}_{\alpha,l} = \hat{I}_0$ at $l_0 = 0$. If $l_0 = \pm 1$ then according to Eqs. (4.42)–(4.44), the operator $\mathbf{s}_{\alpha,l}(z)$ is defined for $z \in \mathcal{P}_b \bigcap \mathbf{C}^\pm$ and acts on $f \in \hat{\mathcal{H}}_0$ as

$$(\mathbf{s}_{\alpha,l}(z)f)(\hat{P}) = \int\limits_{S^2} d\hat{k}' s_\alpha(\hat{k}_\alpha, \hat{k}'_\alpha, z\cos^2\omega) f(\cos\omega_\alpha \hat{k}'_\alpha, \sin\omega_\alpha \hat{p}_\alpha), \tag{6.2}$$

where $\hat{P} = \{\cos\omega_\alpha \hat{k}_\alpha,\ \sin\omega_\alpha \hat{p}_\alpha\}$ and s_α is the scattering matrix (2.16) for the pair subsystem α. We take into account here the fact that $l_0 \cdot \operatorname{Sign}\operatorname{Im} z = 1$ for $l_0 = 1$ as well as $l_0 = -1$. Remember that for $l_0 = 1$ the sheet Π_l is actually the upper half–plane $\mathbf{C}^+$ and for $l_0 = -1$, the lower one, $\mathbf{C}^+$ (in accordance with our choice in Sec. 5 of the part $\Re$ of the total three–body Riemann surface). Therefore one can see now that on the both three–body sheets Π_l, $l_0 = \pm 1$, the operators $\mathbf{s}_{\alpha,l}$ are described by the same formula (6.2). As a matter of fact, the operators $\mathbf{s}_{\alpha,l}(z)$ represent the scattering matrix (2.16) for the pair subsystem α rewritten in the three–body momentum space.

It follows immediately from Eq. (6.2) that if $z \in \mathcal{P}_b \bigcap \mathbf{C}^\pm \setminus Z^{(\alpha)}_{\mathrm{res}}$ then there exists the bounded inverse operator $\mathbf{s}^{-1}_{\alpha,l}(z)$,
$(\mathbf{s}^{-1}_{\alpha,l}(z)f)(\hat{P}) = \int\limits_{S^2} d\hat{k}' s^{-1}_\alpha(\hat{k}_\alpha, \hat{k}'_\alpha, z\cos^2\omega_\alpha) f(\cos\omega_\alpha \hat{k}'_\alpha, \sin\omega_\alpha \hat{p}_\alpha)$ where $s^{-1}_\alpha(\hat{k}, \hat{k}', \zeta)$ stands for the kernel of the inverse pair scattering matrix $s_\alpha(\zeta)$.

The operator $\mathbf{s}^{-1}_{\alpha,l}(z)$ becomes unbounded one at the boundary points z belonging to rims of the cuts ("resonance" rays) included in $Z^{(\alpha)}_{\mathrm{res}}$.

Theorem 8. *The absolute terms $\mathbf{t}_\alpha(P, P', z)$ and kernels $(\mathbf{t}_\alpha R_0)(P, P', z)$ of the Faddeev equations (2.4) admit the analytical continuation in a sense of distributions over $\mathcal{O}(\mathbf{C}^6)$ both on two–body and three–body unphysical sheets Π_l of the surface $\Re$. The continuation on the sheet Π_l with $l = (l_0, l_{1,1}, ..., l_{1,n_1}, l_{2,1}, ..., l_{2,n_2}, l_{3,1}, ..., l_{3,n_3})$, $\quad l_0 = 0$, $l_{\beta,j} = 0, 1$, or $l_0 = \pm 1$, $l_{\beta,j} = 1$ (in both cases $\beta = 1, 2, 3,\quad j = 1, 2, ..., n_\beta$) read as*

$$\mathbf{t}^l_\alpha(z) \equiv \left.\mathbf{t}_\alpha(z)\right|_{\Pi_l} = \mathbf{t}_\alpha - L_0 A_0 \mathbf{t}_\alpha \mathrm{J}_0^\dagger \mathbf{s}^{-1}_{\alpha,l} \mathrm{J}_0 \mathbf{t}_\alpha - \Phi_\alpha \mathrm{J}^{(\alpha)t} L^{(\alpha)} A^{(\alpha)} \mathrm{J}^{(\alpha)} \Phi^*_\alpha, \tag{6.3}$$

$$\left.[\mathbf{t}_\alpha(z) R_0(z)]\right|_{\Pi_l} = \mathbf{t}^l_\alpha(z) R^l_0(z) \tag{6.4}$$

where $R_0^l(z) \equiv \left. R_0(z)\right|_{\Pi_l} = R_0(z) + L_0 A_0(z) \mathrm{J}_0^\dagger(z) \mathrm{J}_0(z)$ *is the continuation [39] on* Π_l *of the free Green function* $R_0(z)$*. If* $l_0 = 0$ *(and consequently,* Π_l *is a two–body sheet) then the continuation (6.3), (6.4) can be made on the hole sheet* Π_l*. For* $l_0 = \pm 1$ *(i.e. in the case if* Π_l *is a three–body sheet) the form (6.3), (6.4) continuation is possible only on the domain* $\mathcal{P}_b \bigcap \Pi_l$*. Al the kernels in r.h. parts of Eqs. (6.3) are taken on the physical sheet.*

Proof of the theorem is based on utilizing the properties of the Cauchy type integrals (see Lemma from Sec. 2 of Ref. [39]), which are the integral terms of Eqs. (2.4).

Using Eqs. (6.3) and (6.4) one can rewrite the Faddeev equations (2.6) continued on the sheet Π_l in the matrix form

$$M^l(z) = \mathbf{t}^l(z) - \mathbf{t}^l(z)\mathbf{R}_0^l(z)\Upsilon M^l(z) \tag{6.5}$$

with

$$\mathbf{t}^l(z) = \mathbf{t} - L_0 A_0 \mathbf{t} \mathbf{J}_0^\dagger \mathbf{s}_l^{-1} \mathbf{J}_0 \mathbf{t} - \Phi \mathrm{J}_1^\dagger L_1 A_1 \mathrm{J}_1 \Phi^*, \tag{6.6}$$

$$\mathbf{R}_0^l(z) = \mathbf{R}_0(z) + L_0 A_0(z) \mathbf{J}_0^\dagger(z) \mathbf{J}_0(z). \tag{6.7}$$

Here, $\mathbf{s}_l(z) = \mathrm{diag}\{\mathbf{s}_{1,l}(z), \mathbf{s}_{2,l}(z), \mathbf{s}_{3,l}(z)\}$. By $M^l(z)$ we denote a supposed analytical continuation of the matrix $M(z)$ on the sheet Π_l.

Theorem 9. *The kernels of the iterations* $\mathcal{Q}^{(n)}(z) = ((-\mathbf{t}\mathbf{R}_0\Upsilon)^n \mathbf{t})(z)$, $n \geq 1$*, allow, in a sense of distributions over* $\mathcal{O}(\mathbf{C}^6)$*, the analytical continuation on the domain* $\Pi_l^{(\mathrm{hol})}$ *of each unphysical sheet* $\Pi_l \subset \Re$*. The continuation is described by* $\left.\mathcal{Q}^{(n)}(z)\right|_{\Pi_l} = ((-\mathbf{t}^l \mathbf{R}_0^l \Upsilon)^n \mathbf{t}^l)(z)$*.*

Remark 2. The products $L_1 \mathrm{J}_1 \Psi^* \Upsilon \mathcal{Q}^{(m)}$, $\mathcal{Q}^{(m)} \Upsilon \Psi \mathrm{J}_1^\dagger L_1$, $\tilde{L}_0 \mathbf{J}_0 \mathcal{Q}^{(m)}$, $\mathcal{Q}^{(m)} \mathbf{J}_0^\dagger \tilde{L}_0$, $L_1 \mathrm{J}_1 \Psi^* \Upsilon \mathcal{Q}^{(m)} \Upsilon \Psi \mathrm{J}_1^\dagger L_1$, $\tilde{L}_0 \mathbf{J}_0 \mathcal{Q}^{(m)} \mathbf{J}_0^\dagger \tilde{L}_0$, $L_1 \mathrm{J}_1 \Psi^* \Upsilon \mathcal{Q}^{(m)} \mathbf{J}_0^\dagger \tilde{L}_0$ and $\tilde{L}_0 \mathbf{J}_0 \mathcal{Q}^{(m)} \Upsilon \Psi \mathrm{J}_1^\dagger L_1$, $0 \leq m < n$, arising at substitution of the relations (6.6) and (6.7) into $\left.\mathcal{Q}^{(n)}(z)\right|_{\Pi_l}$, have to be understood in a sense of the definitions of Sec. 4.

Remark 3. Theorem 9 means that one can pose the continued Faddeev equations (6.5) only in domains $\Pi_l^{(\mathrm{hol})} \subset \Pi_l$.

Construction of the representations for $M^l(z)$ consists actually in an explicit "solving" the continued Faddeev equations (6.5) in the same way as in Refs. [39], [40] where the type (3.2) explicit representations had been found for analytical continuation of T–matrix on unphysical sheets of the energy Riemann surface in the multichannel scattering problem with binary channels. Utilizing the expressions (6.6) for $\mathbf{t}^l(z)$ and (6.7) for $\mathbf{R}_0^l(z)$, we begin with transfer of all the summands including $M^l(z)$ without $\mathbf{J}_0$ and J_1 to the left–hand part of (6.5). Then [for $z \notin \sigma(H)$] we inverse the operators $\mathbf{I} + \mathbf{t}(z)\mathbf{R}_0(z)\Upsilon$, using the relation $(\mathbf{I} + \mathbf{t}\mathbf{R}_0\Upsilon)^{-1} = \mathbf{I} - M\Upsilon\mathbf{R}_0$ (see Ref. [29]). Introducing the new unknowns

$$\begin{aligned} \mathbf{X}_0^{(l)} &= |L_0| \mathbf{s}_l^{-1} \mathbf{J}_0 (\mathbf{I} - \mathbf{t}\mathbf{R}_0) \Upsilon M^l, \\ \mathbf{X}_1^{(l)} &= -L_1 \left[\mathrm{J}_1 \Phi^* \mathbf{R}_0 + A_0 L_0 \mathrm{J}_1 \Phi^* \mathbf{J}_0^\dagger \mathbf{J}_0 \right] \Upsilon M^l, \end{aligned}$$

we obtain for them a closed system of equations which was succeeded to solve explicitly. Expressing then $M^l(z)$ by $\mathbf{X}_0^{(l)}$ and $\mathbf{X}_1^{(l)}$ one comes to the desired representations for $M^l(z)$.

Theorem 10. *The matrix $M(z)$ admits in a sense of distributions over $\mathcal{O}(\mathbf{C}^6)$, the analytical continuation in z on the domains $\Pi_l^{(\mathrm{hol})}$ of unphysical sheets Π_l of the surface $\Re$. The continuation is described by*

$$M^l = M - \left(M\Omega^\dagger \mathrm{J}_0^\dagger, \quad \Phi \mathrm{J}_1^\dagger + M\Upsilon\Psi\mathrm{J}_1^\dagger\right) LA\ S_l^{-1}\tilde{L} \begin{pmatrix} \mathrm{J}_0\Omega M \\ \mathrm{J}_1\Psi^*\Upsilon M + \mathrm{J}_1\Phi^* \end{pmatrix} \tag{6.8}$$

where $S_l(z)$ is the truncated scattering matrix (4.19), $L = \mathrm{diag}\{l_0, l_{1,1}, ..., l_{1,n_1}, l_{2,1}, ..., l_{2,n_2}, l_{3,1}, ..., l_{3,n_3}\}$ and $\tilde{L} = \mathrm{diag}\{|l_0|, l_{1,1}, ..., l_{1,n_1}, l_{2,1}, ..., l_{2,n_2}, l_{3,1}, ..., l_{3,n_3}\}$. Kernels of all the operators in the right–hand part of Eq. (6.8) are taken on the physical sheet.

Note that $LA\ S_l^{-1}(z)\tilde{L} = \tilde{L}[S_l^\dagger(z)]^{-1}\ AL$. Thus, the relations (6.8) may be rewritten also in terms of the scattering matrices $S_l^\dagger(z)$. It is clear that these relations may be rewritten in terms of symmetrized (truncated) scattering matrices [28], too.

The representations for continuation of the (truncated) scattering matrices $S_l(z)$, $S_l(z) : \hat{\mathcal{H}}_0 \oplus \hat{\mathcal{H}}_1 \to \hat{\mathcal{H}}_0 \oplus \hat{\mathcal{H}}_1$ and $S_l^\dagger(z)$, $S_l^\dagger(z) : \hat{\mathcal{H}}_0 \oplus \hat{\mathcal{H}}_1 \to \hat{\mathcal{H}}_0 \oplus \hat{\mathcal{H}}_1$, follow from the representations (6.8) for $M^l(z)$. Before writing final formulae we make some remarks.

First of all, we note that the function $A_0(z)$ is univalent. It looks as $A_0(z) = -\pi i z^2$ on all the sheets Π_l. At the same time after continuing from Π_0 on Π_l, the function $A_{\beta,j}(z) = -\pi i\sqrt{z-\lambda_{\beta,j}}$ keeps its form if only $l_{\beta,j} = 0$. If $l_{\beta,j} = 1$ this function turns into $A'_{\beta,j}(z) = -A_{\beta,j}(z)$. Analogous inversion takes (or does not take) place for the arguments $\hat{P}$, $\hat{P}'$, $\hat{p}_\alpha$ and $\hat{p}'_\beta$ of kernels of the operators $\mathrm{J}_0\Omega M\Omega^\dagger \mathrm{J}_0^\dagger$, $\mathrm{J}_0\Omega M\Upsilon\Psi\mathrm{J}_1^\dagger$, $\mathrm{J}_1\Psi^*\Upsilon M\Omega^\dagger\mathrm{J}_0^\dagger$ and $\mathrm{J}_1\Psi^*(\Upsilon\mathbf{v}+\Upsilon M\Upsilon)\Psi\mathrm{J}_1^\dagger$, too. Remember that on the physical sheet Π_0, the action of $\mathrm{J}_0(z)$ $(\mathrm{J}_0^\dagger(z))$ transforms $P \in \mathbf{R}^6$ in $\sqrt{z}\hat{P}$ $(P' \in \mathbf{R}^6$ in $\sqrt{z}\hat{P}')$. At the same time, $p_\alpha \in \mathbf{R}^3$ $(p'_\beta \in \mathbf{R}^3)$ turns under $\mathrm{J}_{\alpha,i}(z)$ $(\mathrm{J}_{\beta,j}^\dagger(z))$ into $\sqrt{z-\lambda_{\alpha,i}}\,\hat{p}_\alpha$ $(\sqrt{z-\lambda_{\beta,j}}\,\hat{p}'_\beta)$. Therefore we introduce the operators $\mathcal{E}(l) = \mathrm{diag}\{\mathcal{E}_0, \mathcal{E}_1\}$ where $\mathcal{E}_0$ is the identity operator in $\hat{\mathcal{H}}_0$ if $l_0 = 0$, and $\mathcal{E}_0$, the inversion $(\mathcal{E}_0 f)(\hat{P}) = f(-\hat{P})$ if $l_0 = \pm 1$. Analogously, $\mathcal{E}_1(l) = \mathrm{diag}\{\mathcal{E}_{1,1}, ..., \mathcal{E}_{1,n_1}; \mathcal{E}_{2,1}, ..., \mathcal{E}_{2,n_2}; \mathcal{E}_{3,1}, ..., \mathcal{E}_{3,n_3}\}$ where $\mathcal{E}_{\beta,j}$ is the identity operator in $\hat{\mathcal{H}}^{(\beta,j)}$ if $l_{\beta,j} = 0$, and $\mathcal{E}_{\beta,j}$, the inversion $(\mathcal{E}_{\beta,j} f)(\hat{p}_\beta) = f(-\hat{p}_\beta)$ if $l_{\beta,j} = 1$. By $\mathrm{e}_1(l)$ we denote the diagonal matrix $\mathrm{e}_1(l) = \mathrm{diag}\{\mathrm{e}_{1,1}, ..., \mathrm{e}_{1,n_1}; \mathrm{e}_{2,1}, ..., \mathrm{e}_{2,n_2}; \mathrm{e}_{3,1}, ..., \mathrm{e}_{3,n_3}\}$ with the elements $\mathrm{e}_{\beta,j} = 1$ if $l_{\beta,j} = 0$ and $\mathrm{e}_{\beta,j} = -1$ if $l_{\beta,j} = 1$. Let $\mathrm{e}(l) = \mathrm{diag}\{\mathrm{e}_0, \mathrm{e}_1\}$ where $\mathrm{e}_0 = +1$.

Theorem 11. *If there exists a path on the surface $\Re$ such that at moving by it from the domain $\Pi_l^{(\mathrm{hol})}$ on Π_0 to the domain $\Pi_l^{(\mathrm{hol})} \bigcap \Pi_{ll'}^{(\mathrm{hol})}$ on $\Pi_{l'}$, the parameter z stays on intermediate sheets $\Pi_{l''}$ always in the domains $\Pi_l^{(\mathrm{hol})} \bigcap \Pi_{ll''}^{(\mathrm{hol})}$, then the truncated scattering matrices $S_l(z)$ and $S_l^\dagger(z)$ admit analytical continuation in z on the domain $\Pi_l^{(\mathrm{hol})} \bigcap \Pi_{ll'}^{(\mathrm{hol})}$ of the sheet $\Pi_{l'}$. The continuation is described by*

$$S_l(z)|_{\Pi_{l'}} = \mathcal{E}(l')\left[\hat{\mathbf{I}} + \tilde{L}\hat{\mathcal{T}}L\ A\mathrm{e}(l') - \tilde{L}\hat{\mathcal{T}}L'\ A\,S_{l'}^{-1}\ \tilde{L}'\hat{\mathcal{T}}L\ A\mathrm{e}(l')\right]\mathcal{E}(l'), \tag{6.9}$$

$$S_l^\dagger(z)\Big|_{\Pi_{l'}} = \mathcal{E}(l')\left[\hat{\mathbf{I}} + \mathrm{e}(l')A\ L\hat{\mathcal{T}}\tilde{L} - \mathrm{e}(l')A\ L\hat{\mathcal{T}}\tilde{L}'\ [S_{l'}^\dagger]^{-1}\ A\ L'\hat{\mathcal{T}}\tilde{L}\right]\mathcal{E}(l') \tag{6.10}$$

where $L' = \left\{l'_0, l'_{1,1}, ..., l'_{1,n_1}, l'_{2,1}, ..., l'_{2,n_2}, l'_{3,1}, ..., l'_{3,n_3}\right\}$ and $\tilde{L}' = \left\{|l'_0|, l'_{1,1}, ..., l'_{1,n_1}, l'_{2,1}, ..., l'_{2,n_2}, l'_{3,1}, ..., l'_{3,n_3}\right\}$.

As we have established, the kernels of all the operators present in the right–hand part of expression (2.7) for the resolvent $R(z)$ admit, in a sense of distributions over $\mathcal{O}(\mathbf{C}^6)$, the analytical continuation on the domains $\Pi_l^{(\mathrm{hol})}$ of unphysical sheets $\Pi_l \subset \Re$. Hence, the kernel $R(P, P', z)$ admits such representation, too.

Theorem 12. *The analytical continuation, in a sense of distributions over $\mathcal{O}(\mathbf{C}^6)$, of the resolvent $R(z)$ on the domain $\Pi_l^{(\mathrm{hol})}$ of unphysical sheet $\Pi_l \subset \Re$ is described by the formula*

$$R(z)|_{\Pi_l} = R+ \\ +([I-RV]\mathrm{J}_0^\dagger\,, \quad \Omega[\mathbf{I}-\mathbf{R}_0 M\Upsilon]\Psi \mathrm{J}_1^\dagger) LAS_l^{-1}\tilde{L} \begin{pmatrix} \mathrm{J}_0[I-VR] \\ \mathrm{J}_1\Psi^*[\mathbf{I}-\Upsilon M\mathbf{R}_0]\Omega^\dagger \end{pmatrix}. \qquad (6.11)$$

Kernels of all the operators present in the right–hand part of Eq. (6.11) are taken on the physical sheet.

Note that in their structure, the representations (6.11) are quite analogous to that for analytical continuation of the two–body resolvent (3.7). Proof of the expressions (6.11) are based on immediate using the representations (6.8) for $M^l(z)$.

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